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APPROXIMATE QUADRATURE.

BY PROF. ASAPH HALL.

(1). The general expression for the area contained between two ordinates, the axis of $x$ and any curve is $\int y \, dx$, the integral being taken between the limits $x = g$, and $x = h$; to which correspond the terminal ordinates. Since it is always possible to change the limits of integration to 0 and 1 by putting $x = g + (h - g) t$, so that

$$\int_{g}^{h} y \, dx = (h - g) \int_{0}^{1} y \, dt,$$

we need consider only the integral $\int_{0}^{1} y \, dt$. Frequently however the integration is impossible, or the function $y$ is unknown, and knowing by observation or otherwise certain values of the ordinates we are obliged to resort to approximate quadrature. The simplest method is this: divide the interval, 0 — 1, into $n$ equal parts and join by right lines the tops of the ordinates drawn at the points of division. We have now $n$ trapezoids formed by the $n + 1$ ordinates, and taking the half sum of the sides for the height of a trapezoid and multiplying by the common base, $A = \frac{1}{n}$, we have the approximate value

$$\int_{0}^{1} y \, dt = A. (\frac{y_{0} + y_{1} + y_{2} + \ldots + y_{n-1}}{2} + \frac{1}{2} y_{n}).$$

This method is obvious and must be very ancient. It is a correct principle of approximation and when $n$ is infinite gives the exact value of the integral.

(2). The next step in the quadrature would probably be this. Instead of joining the tops of the ordinates by right lines assume some curve joining them. The general parabolic curve

$$y = a_{0} + a_{1}x + a_{2}x^{2} + \ldots + a_{n}x^{n},$$
can be made to pass through \( n + 1 \) points, there being \( n + 1 \) constants at our disposal. Let us take a quadratic parabola, which can be made to pass through three points. Its axis being parallel to the ordinates, and assuming the axis of \( x \) tangent to the vertex, the equation of the parabola is \( y = \frac{x^2}{p} \).

Hence the area between the axis of \( x \), the ordinate and the curve is \( \frac{x^3}{3p} \): the arbitrary constant being zero. The area between the first and third ordinates, the axis of \( x \) and the curve is

\[
\frac{(x + 2A)^3 - x^3}{3p} = \frac{A}{3} \cdot \frac{6x^2 + 12Ax + 8A^2}{p} = \frac{A}{3} \left[ \frac{x^2}{p} + 4 \frac{(x + A)^3}{p} + \frac{(x + 2A)^3}{p} \right],
\]

or

\[
\text{Area} = \frac{1}{3} \frac{A}{4} \left( y_0 + 4y_1 + y_2 \right).
\]

This result has been obtained by supposing the axis of \( x \) tangent to the vertex of the parabola, but a similar formula is found if we move the axis of \( x \) downward a distance \( = b \). The area will then be

\[
2bA + \frac{1}{3} A \left[ y_0 - b + 4(y_1 - b) + y_2 - b \right] = \frac{1}{3} A \left( y_0 + 4y_1 + y_2 \right),
\]

in which the ordinates are measured from the original axis of \( x \). If we apply this formula to the groups \( y_0y_1y_2 \) : \( y_3y_4y_5 \) : \( \ldots \) : \( \ldots y_n \) : \&c., we have

\[
\int ydx = \frac{1}{3} A \left[ y_0 + y_1 + 2(y_2 + y_3 + y_4 + \ldots) + 4(y_1 + y_3 + \ldots) \right].
\]

This is known as Simpson's rule, and is in common use for computing the approximate values of areas; also the tonnage of vessels, the capacity of casks, \&c. In determining the solid content \( y \) is the surface of the section made by a plane perpendicular to the axis of \( x \).

(3). The question of quadradure like that of interpolation is indeterminate and admits of an indefinite number of methods. Whatever the function \( y \) may be if it is continuous we can express the general value of an ordinate by means of the known theorem for the expansion of a function, and since the ordinates are equidistant we have by the calculus of finite differences

\[
y = y_0 + xA_0 + \frac{x(x-1)}{1.2} A^2y_0 + \frac{x(x-1)(x-2)}{1.2.3} A^3y_0 + \frac{x(x-1)(x-2)(x-3)}{1.2.3.4} A^4y_0 + \&c.
\]

This is the equation of a parabolic curve like that assumed in (2), since \( A_0, A_2y_0, \&c. \) are constants. Multiply by \( dx \) and integrate from 0 to 1. Then

\[
\int ydx = y_0 + \frac{A}{12} y_0 + \frac{1}{24} A^2y_0 - \frac{19}{720} A^3y_0 + \frac{3}{160} A^4y_0 + \&c.,
\]

which will be the area comprised between the ordinates \( y_0 \) and \( y_1 \). In the same way the area comprised between \( y_1 \) and \( y_2 \) will be
\[ \int y \, dx = y_1 + \frac{1}{2} \Delta y_1 - \frac{1}{12} \Delta^2 y_1 + \frac{1}{24} \Delta^3 y_1 - \&c., \]

and similar values for the other sections. Adding these integrals and observing that
\[ \Delta y_0 + \Delta y_1 + \cdots + \Delta y_{n-1} = y_1 - y_0 + y_2 - y_1 + \cdots + y_n - y_{n-1} = y_n - y_0, \]
\[ \Delta^2 y_0 + \Delta^2 y_1 + \cdots + \Delta^2 y_{n-1} = \Delta^2 y_n - \Delta^2 y_0 + \&c., \]
we have for the value of the integral between the extreme ordinates, after some slight reductions, the formula given by Laplace for computing the perturbations of a comet, and which is generally given in French and English books. The first term is the sum of the trapezoids which is corrected by the following terms depending on the differences. This formula is not however so convenient as the one given by Gauss, see Werke Vol. III, p.3 28.

(4). If we multiply the preceding general value of \( y \) by \( dx \) and integrate between the limits 0 and \( n \), where \( n \) denotes the number of intervals, we have

\[ \int_0^n y \, dx = ny_0 + \frac{n^2}{2} \Delta y_0 + \left( \frac{n^3}{6} - \frac{n^2}{4} \right) \Delta^2 y_0 + \left( \frac{n^4}{24} - \frac{n^3}{6} + \frac{n^2}{6} \right) \Delta^3 y_0 \]
\[ + \left( \frac{n^5}{120} - \frac{n^4}{16} + \frac{11n^3}{8} - \frac{n^2}{12} \right) \Delta^4 y_0 + \left( \frac{n^6}{720} - \frac{n^5}{60} + \frac{7n^4}{96} - \frac{5n^3}{36} + \frac{n^2}{10} \right) \Delta^5 y_0 \]
\[ + \left( \frac{n^7}{5040} - \frac{n^6}{288} + \frac{17n^5}{512} - \frac{5n^4}{64} + \frac{137n^3}{1080} - \frac{n^2}{12} \right) \Delta^6 y_0 + \&c. \]

If we put \( n = 2 \), and neglect the differences above \( \Delta \), we have, since \( \Delta y_0 = y_1 - y_0 \), and \( \Delta^2 y_0 = y_2 - 2y_1 + y_0 \),

\[ \int_0^2 y \, dx = \frac{1}{3} \left( y_0 + 4y_1 + y_2 \right), \]

and the application of this formula to the whole interval will give Simpson's rule. Putting \( n = 3 \), we have the formula given by Newton,

\[ \int_0^3 y \, dx = \frac{3}{8} \left( y_0 + 3y_1 + 3y_2 + y_3 \right), \]

and which Cotes calls the \textit{pulcherrima et utilisima regula}.

If \( n = 6 \), we have

\[ \int_0^6 y \, dx = 6y_0 + 18 \Delta y_0 + 27 \Delta^2 y_0 + 24 \Delta^3 y_0 + \frac{123}{10} \Delta^4 y_0 + \frac{33}{10} \Delta^5 y_0 + \frac{41}{140} \Delta^6 y_0 + \&c. \]

As \( \Delta^6 y_0 \) is generally a small quantity, instead of \( \Delta^6 y_0 \), put \( \Delta^6 y_0 = 1 \), and then reducing by the formula \( \Delta y_0 = y_1 - y_0 \), \&c. and denoting the common interval by \( \theta \) we have the convenient and accurate formula given by Mr. Weddle,
\[ \int_0^\theta ydx = \frac{3\theta}{10} \left[ y_0 + y_2 + y_4 + y_6 + 5 \left( y_1 + y_3 + y_5 \right) + y_8 \right]. \]

This formula is designed for six intervals, or seven ordinates, and we have to take the sum of the even ordinates and the middle ordinate, add to this five times the sum of the odd ordinates, and multiply the whole sum by three tenths of the common interval.

(5). The following elegant method was given by Roger Cotes in his *Harmonia Mensurarum*, published in 1722, and is an elaboration of the method indicated by Newton. In the expression \( ydt \) substitute for \( y \) an entire function of \( t \) which shall agree with all the given values of \( y \). If

\[ T_m = \frac{nt(nt - 1)(nt - 2)(nt - 3)\ldots(nt - n)}{nt - m}, \]

the function

\[ \sum_{m=0}^{\infty} y_m \cdot \frac{T_m}{M_m} \]

reduces to \( y_m \) for the value \( nt = m \), and has \( n + 1 \) values common with \( y \). As \( T_m \) and \( M_m \) are functions of \( t \) only we can compute the integral

\[ \int_0^1 \frac{T_m}{M_m} dt = A_m, \]

and then shall have

\[ \int_0^1 ydt = A_0y_0 + A_1y_1 + A_2y_2 + \ldots + A_ny_n, \]

the values of \( A_0, A_1, A_2, \&c., \) serving for all cases. The values of the ratio \( \frac{T_m}{M_m} \) are

\[ \frac{T_0}{M_0} = \frac{(nt - 1)(nt - 2)(nt - 3)\ldots(nt - n)}{(-1)(-2)(-3)\ldots(-n)}, \]

\[ \frac{T_1}{M_1} = \frac{nt(nt - 2)(nt - 3)\ldots\ldots\ldots(nt - n)}{1(-1)(-2)\ldots(1-n)}, \]

\[ \frac{T_2}{M_2} = \frac{nt(nt - 1)(nt - 3)\ldots\ldots\ldots(nt - n)}{2.1(-1)\ldots(2-n)}, \]

\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]

\[ \frac{T_n}{M_n} = \frac{nt(nt - 1)(nt - 2)\ldots\ldots(nt - n + 1)}{n.(n - 1)(n - 2)\ldots\ldots 1}. \]

As an example we compute the value of \( A_2 \) for \( n = 4 \), or for 5 ordinates. In this case we have

\[ T_2 = 4t(4t - 1)(4t - 3)(4t - 4) = 256t^4 - 512t^3 + 304t^2 - 48t, \]

\[ M_2 = 2.1.(-1)(-2) = +4. \]

Integrating the value of \( T_2 \) from 0 to 1 we have
\[ 4A_2 = \frac{256}{5} - \frac{512}{4} + \frac{304}{3} - \frac{48}{2} = + \frac{8}{15}, \]

or

\[ A_2 = + \frac{2}{15}. \]

Cotes has computed these coefficients for all cases to \( n = 10 \), or for eleven ordinates. The following are the values to \( n = 6 \).

For \( n = 1 \), \( A_0 = A_1 = \frac{1}{2} \),

For \( n = 2 \), \( A_0 = A_2 = \frac{1}{4} \), \( A_1 = \frac{3}{8} \),

For \( n = 3 \), \( A_0 = A_3 = \frac{1}{3} \), \( A_1 = A_2 = \frac{3}{8} \),

For \( n = 4 \), \( A_0 = A_4 = \frac{7}{16} \), \( A_1 = A_2 = \frac{1}{4} \), \( A_3 = \frac{3}{8} \),

For \( n = 5 \), \( A_0 = A_5 = \frac{19}{288} \), \( A_1 = A_4 = \frac{33}{64} \), \( A_2 = A_3 = \frac{5}{16} \),

For \( n = 6 \), \( A_0 = A_6 = \frac{41}{840} \), \( A_1 = A_5 = \frac{3}{35} \), \( A_2 = A_4 = \frac{3}{28} \), \( A_3 = \frac{3}{16} \).

To find the approximate value of the integral by this method we have only to multiply the known value of the ordinate by the corresponding value of \( A \) and take the sum of the products. It will be noticed that we always have \( A_{n-m} = A_m \).

(6). The preceding formule are in such form that it is not necessary to regard the differences of the ordinates, but it is best in applying such methods to avoid large and irregular differences, and if necessary to diminish the interval for which the ordinates are computed. To show by example the approximation of the formula let it be required to compute the area bounded by the two ordinates drawn midway between the center and circumference of a circle, the diameter and the curve. Taking the radius as unit, the interval is \( \frac{1}{4} \), and the values of the seven equidistant ordinates are,

\[
y_0 = y_6 = \frac{1}{2} \sqrt{3} = 0.8660254
\]

\[
y_1 = y_5 = \frac{1}{4} \sqrt{8} = 0.9428090
\]

\[
y_2 = y_4 = \frac{1}{4} \sqrt{35} = 0.9860133
\]

\[
y_3 = 1.
\]

The exact value of this area is

\[
2 \int_0^{\frac{1}{4}} \sqrt{1 - x^2} \, dx = \left[ x \sqrt{1 - x^2} + \sin^{-1} x \right]_0^{\frac{1}{4}} = \frac{\sqrt{3}}{4} + \frac{\pi}{6} = 0.9566115.
\]

By trapezoids area = 0.9539450 error = –0.0026655,

" Simpson’s rule " = 0.9565876 " = –0.0000239,

" Weddle’s rule " = 0.9566084 " = –0.0000031,

" Cotes’ method " = 0.9566099 " = –0.0000016.

Under certain restrictions the preceding formule become exact. Thus Simpson’s rule gives the exact integral when the bounding curve is a quadratic parabola, and the exact cubature when the solid is bounded by certain parabolic surfaces. There is a peculiarity in the approximation by Cotes’ method which is worthy of notice. It is that in passing from an odd number
of ordinates to an even number the degree of the error of the quadrature does not change although its amount is diminished, but when we pass from an even to an odd number the error is diminished two degrees. It is therefore better to use an odd number of ordinates.

(7). Hitherto we have assumed that the line of abscissas over which the quadrature is extended is divided into equal parts, and that $y$, or function of $x$, is expressed by a converging series and as an entire polynomial of the $n^{th}$ degree, so that

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n.$$

If the function is an entire polynomial of the $n^{th}$ degree, or less, the quadrature is exact, but if there are other terms, such as $a_{n+1} x^{n+1}$, &c., the error will be produced by these terms and will be the sum of the integrals $\int a_{n+1} x^{n+1} dx$, &c. In 1814 Gauss published a memoir on approximate quadrature, probably having been led to consider the subject from its use in computing the perturbations of the minor planets. The title of this memoir is "Methodus nova Integralium Valores per approximationem inveniendi." Werke, Band 3, p. 165. Gauss lays aside the restriction of an equal division of the line of abscissas and shows that by a suitable division of this line the degree of approximation can be nearly doubled. His method consists in choosing the abscissas in such a way that the polynomial may be of the $2n^{th}$ degree and the error still remains zero.

(8). Let us take a function of $x$ which shall produce the given values of $y$ when $x$ has certain values. The form which Lagrange employs in his method of interpolation gives, denoting by $a$, $a_1$, $a_2$, &c. the given values of $x$,

$$y = y_0 \cdot \frac{(x - a_1)(x - a_2)(x - a_3)\ldots(x - a_n)}{(a - a_1)(a - a_2)(a - a_3)\ldots(a - a_n)} + y_1 \cdot \frac{(x - a)(x - a_1)(x - a_2)\ldots(x - a_n)}{(a - a)(a_1 - a)(a_2 - a)\ldots(a_1 - a_n)} + y_2 \cdot \frac{(x - a)(x - a_1)(x - a_2)\ldots(x - a_n)}{(a_2 - a)(a_2 - a_1)(a_2 - a_3)\ldots(a_2 - a_n)} + \ldots + y_n \cdot \frac{(x - a)(x - a_1)(x - a_2)\ldots(x - a_{n-1})}{(a_n - a)(a_n - a_1)(a_n - a_2)\ldots(a_n - a_{n-1})}.$$

This function evidently gives $y = y_0$, $y = y_1$, &c. for $x = a$, $x = a_1$, &c. Let

$$X = (x - a)(x - a_1)(x - a_2)\ldots(x - a_n)$$

$$= x^{n+1} + c_1 x^n + c_2 x^{n-1} + \ldots + c_n.$$

The numerators in the value of $y$ are $\frac{X}{x - a}$, $\frac{X}{x - a_1}$, $\frac{X}{x - a_2}$, &c., and
the denominators are the values of the numerators when \( x = a, x = a_1, \&c., \) and if we denote them by \( M_0, M_1, M_2, \&c., \) then
\[
y = \frac{X}{(x-a)M_0} \cdot y_0 + \frac{X}{(x-a_1)M_1} \cdot y_1 + \frac{X}{(x-a_2)M_2} \cdot y_2 + \ldots + \frac{X}{(x-a_n)M_n} \cdot y_n.
\]
Since \( X = 0, \) for \( x = a, \) we have
\[
a^{n+1} + ca^n + c_1a^{n-1} + \ldots + c_n = 0,
\]
and
\[
X = x^{n+1} - a^{n+1} + c(x^n - a^n) + c_1(x^{n-1} - a^{n-1}) + \ldots + c_{n-1}(x - a),
\]
and dividing by \( x - a, \)
\[
\frac{X}{x-a} = x^n + ax^{n-1} + a^2x^{n-2} + \ldots + a^n,
\]
and
\[
+ ca^{n-1} + c_1a_{n-2} + c_2a^{n-3} + \ldots + c_{n-1}a^n,
\]
\[
+ c_1x^{n-2} + c_2x^{n-3} + \ldots + c_{n-1}x^n,
\]
\[
+ \ldots + c_{n-1}.
\]
For \( x = a, \) this gives
\[
\frac{X}{x-a} = (n+1)a^n + nca^{n-1} + (n-1)c_1a^{n-2} + \ldots + c_{n-1}.
\]
Hence \( M_0 \) is the value of \( \frac{dX}{dx} \) for \( x = a, \) and likewise \( M_1, M_2, \&c., \) are the values for \( x = a_1, x = a_2, \&c. \)

We have also
\[
\int_0^1 \frac{Xdx}{x-a} = \frac{1}{n+1} + \frac{a}{n} + \frac{a^2}{n-1} + \ldots + \frac{a^n}{c}
\]
\[
+ \frac{ca}{n-1} + \ldots + c_1a^{n-2}
\]
\[
+ \frac{c}{n-2}a^{n-3} + \ldots + c_2a^{n-3}
\]
\[
+ \ldots + c_{n-1}.
\]

If we write these terms in the following order
\[
a^n + ca^{n-1} + c_1a^{n-2} + c_2a^{n-3} + \ldots + c_{n-1}
\]
\[
+ \frac{1}{2}(a^{n-1} + c'a^{n-2} + c_1a^{n-3} + \ldots + c_{n-2})
\]
\[
+ \frac{1}{3}(a^{n-2} + c'a^{n-3} + c_1a^{n-4} + \ldots + c_{n-3})
\]
\[
+ \ldots + \ldots + \ldots + \frac{1}{n}(a + c)
\]
\[
+ \frac{1}{n+1}.
\]
it is evident that the same quantity will be produced if in the product arising from the multiplication of the series for $X$ by the infinite series

$$x^{-1} + \frac{1}{2}x^{-2} + \frac{1}{3}x^{-3} + \frac{1}{4}x^{-4} + \ldots = \log \frac{x}{x-1},$$

we reject all terms containing negative powers of $x$ and write $a$ for $x$. If therefore we put

$$\int_0^1 \frac{Xdx}{x-a} = X', \text{ for } x = a,$$

and denote the values of the function $\frac{X'}{dX/dx}$ for $x = a, x = a_1, x = a_2, \&c.$ by $A_0, A_1, A_2, \&c.$, we shall have for the approximate value of the integral,

$$\int_0^1 ydx = A_0 y_0 + A_1 y_1 + A_2 y_2 + \ldots A_n y_n.$$

Hence we have the means of computing the coefficients $A_0, A_1, A_2, \&c.$, when the values of $x$ are given equal to $a, a_1, a_2, \&c.$

(9). We have now to determine the values of $x$ so that the polynomial which expresses the value of $y$ may rise to the $2n^{th}$ degree and the error of quadrature be zero at the same time.

We have $y = f(x)$, and let

$$\varphi(x) = (x - a)(x - a_1)(x - a_2) \ldots (x - a_n).$$

As $f(x)$ is a polynomial of the $2n^{th}$ degree, or less, if we denote by $Q$ the quotient of this polynomial divided by $\varphi(x)$ and by $R$ the remainder we have

$$f(x) = Q\varphi(x) + R.$$  

We now assume $R$ for $f(x)$, since for the values $a, a_1, a_2, \&c., \varphi(x)$ is zero, and the error in the quadrature will be $\int_0^1 Q\varphi(x)dx$. Since $Q$ does not exceed the degree $n-1$ in order that the error may be zero we determine $\varphi(x)$ by the conditions,

$$\int_0^1 \varphi(x)dx = 0, \quad \int_0^1 x\varphi(x)dx = 0, \quad \ldots \quad \int_0^1 x^{n-1}\varphi(x)dx = 0.$$  

By means of a known formula of reduction, or by integrating by parts, the integral $\int x^n\varphi(x)dx$ can be reduced to multiple integrals of $\varphi(x)$. Thus

$$\int \varphi(x)dx = \int \varphi(x)dx,$$

$$\int x\varphi(x)dx = x\int \varphi(x)dx - \int \varphi(x)dx^2,$$

$$\int x^2\varphi(x)dx = x^2\int \varphi(x)dx - 2x\int \varphi(x)dx^2 + 2\int \varphi(x)dx^3,$$

$$\int x^n\varphi(x)dx = x^{n-1}\int \varphi(x)dx - (m-1)x^{n-2}\int \varphi(x)dx^2 + (m-1)(m-2) \times x^{n-3}\int \varphi(x)dx^3 + \ldots + 1\int \varphi(x)dx^n.$$
Hence in order that \( \int \varphi(x)dx, \int x \varphi(x)dx, \ldots \) become zero between the limits 0 and 1 the integrals \( \int \varphi(x)dx, \int x^2 \varphi(x)dx, \ldots \) must vanish for the same limits. We have therefore to find a function such that the function itself and its 1st, 2nd, 3rd, and nth differentials vanish for \( x = 0 \), and \( x = 1 \). If \( \Pi \varphi \) be the function sought \( \Pi \varphi \) must have the factors \( x^{n+1} \) and \( (x-1)^{n+1} \), and inversely every function which has the factor \( x^{n+1}(x-1)^{n+1} \), and besides only constant factors, fulfills the condition. Since \( \varphi(x) \) is of the \( (n+1)\)th order, \( \Pi \varphi = \int x^{n+1} \varphi(x)dx \), is of the \( (2n+2)\)th order. If we put \( \Pi \varphi = x^{n+1}(x-1)^{n+1} \cdot M \), where \( M \) is a constant, we have

\[
\varphi(x) = \frac{M \cdot x^{n+1}(x-1)^{n+1}}{dx^{n+1}}.
\]

Applying the rule for the differentiation of the product of two functions of \( x \) and multiplying by

\[
M = \frac{1}{(2n+2)(2n+1)(2n)\ldots(n+2)}
\]

\[
\varphi(x) = x^{n+1} - \frac{(n+1)^2}{2n+2} x^n + \frac{(n+1)^2 x^2}{1 \cdot 2 \cdot (2n+2)(2n+1)} x^{n-1} - \frac{(n+1)^2 n^2 (n-1)^2}{1 \cdot 2 \cdot 3 \cdot (2n+2)(2n+1)(2n)} x^{n-2} \quad \& \& \\
+ \frac{(-1)^{n+1}}{(2n+2)(2n+1)(2n)(2n-1)\ldots n+2} (n+1)(n-1)(n-2) \ldots \ldots \ldots \ldots \ldots \ldots 1
\]

If now we assume for \( n \) the values 0, 1, 2, 3, 4, \&c., and reject terms containing negative powers of \( x \) the roots of this equation will give the values of the abscissas for the Gaussian method. These roots are all real and lie between 0 and 1, and when there is an odd number the middle one is \( \frac{1}{2} \).

The approximate value of the integral is therefore

\[
\int_0^1 ydx = A_0 \varphi(a) + A_1 \varphi(a_1) + A_2 \varphi(a_2) + \ldots + A_n \varphi(a_n)
\]

The following are the values of the roots and coefficients for \( n = 0, n = 1, n = 2, \) and \( n = 3 \).

For \( n = 0 \)

\[
\begin{align*}
\alpha &= 0.5 \\
A_0 &= 1
\end{align*}
\]

For \( n = 1 \)

\[
\begin{align*}
\alpha &= 0.21132487 \\
A_1 &= 0.78867513 \\
A_0 &= \frac{1}{2}
\end{align*}
\]

For \( n = 2 \)

\[
\begin{align*}
\alpha &= 0.11270167 \\
a_1 &= 0.5 \\
a_2 &= 0.88729833 \\
A_0 &= \frac{1}{3} \\
A_1 &= \frac{1}{3} \\
A_2 &= \frac{1}{3}
\end{align*}
\]

For \( n = 3 \)

\[
\begin{align*}
\alpha &= 0.06943184 \\
a_1 &= 0.33000948 \\
a_2 &= 0.66999052 \\
A_2 &= 0.32607258 \\
a_3 &= 0.93056816 \\
A_3 &= 0.17392742
\end{align*}
\]
Gauss has computed these roots and coefficients for all values of \( n \), from \( n = 0 \), to \( n = 6 \), to sixteen places of decimals; and a calculation of the errors shows that the degree of approximation as compared with the Cotesian method is nearly doubled. Computing the area of article (6) by the Gaussian method, and putting \( n = 3 \), or using four ordinates, I find
\[
\log \varphi(a) = \log \varphi(a_3) = 9.9554751, \quad \log \varphi(a_1) = \log \varphi(a_2) = 9.9936327,
\]
\[
\log A_0 = 9.2403681, \quad \log A_1 = 9.5133143.
\]
Hence we have,
\[
A_0 \varphi(a) = 0.1569796 \\
A_1 \varphi(a_1) = 0.3213268 \\
A_2 \varphi(a_2) = 0.3213268 \\
A_3 \varphi(a_3) = 0.1569796
\]
\[
\therefore \quad \text{Area} = 0.9566128, \text{and error} = + 0.0000013.
\]

The Gaussian method has not, so far as I know, been brought much into use, and there are reasons for this in astronomical calculations, since in computing anomalies, radii vectores, &c., the intervals of time being equal, the method of differences gives us a valuable check on the numerical work. But it seems to me that this method may be advantageously applied to series of observations. Thus if one wishes to determine the barometric or thermometric curve for any place, and takes the time the sun is above the horizon for the interval of observation, and decides to make three observations a day, for which \( n = 2 \), he will observe his instruments about one ninth of the interval after sunrise and before sunset, and at noon. Hence he will get as good a result as from five observations equally distributed over the interval. In the course of a year even the saving of labor in observing and reducing would be very great. While the need of a judicious arrangement of the observations is more apparent in meteorology, there can be no doubt that this question will at length be considered in astronomical observations; since in astronomy the objects to be observed are constantly increasing in number, and this increase will necessitate a more careful expenditure of the labor of the observer.

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**EXTRACTION OF ROOTS.**

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**BY ALEXANDER EVANS, ELKTON, MARYLAND.**

Let \( N \) = the number whose root is to be extracted, \( r \) = an approximate root, \( \frac{1}{n} \) = the degree of the root and \( R \) = the true root. Then, approximately,
\[
\frac{N}{n^{r-1}} + \frac{n-1}{n} r = R. \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1)
\]

This formula is easily remembered, is applicable to roots of any degree, always converges, and rapidly; and to practical men, who often forget how to calculate a cube, and sometimes a square root, will be of value in the absence of a table of logarithms.

Let it be required to find the square root of 2: suppose \( r = 1.4 \): Then

\[
\frac{2}{2(1\cdot4)} + \frac{1}{2} \cdot \frac{14}{10} = R; \text{ i.e. } \frac{10}{14} + \frac{14}{20} = \sqrt{2} = \frac{99}{70} = 1.4142,
\]

true to four places. By substituting \( \frac{9}{10} \) in the formula, we have

\[
\frac{70}{99} + \frac{99}{140} = R = \frac{19601}{13860}, \text{ true to eight places. If } \frac{19601}{13860} \text{ be substituted, there results }
\]

\[
\frac{13860}{19601} + \frac{19601}{27720}, \text{ true to eighteen places: viz., } \sqrt{2} = 1.4142135623730950499; \text{ the true final figures being 488: see De Morgan, Budget of paradoxes, page 293; Analyst, Vol. I, page 119.}
\]

In this particular example the successive fractions are easily formed by taking the denominator of the first for the numerator of the second fraction, and double the numerator of the first for the denominator of the second. By proceeding in this manner, the value of \( \sqrt{2} \) may, in a short time and with comparative ease, be found to one hundred places. This method therefore, will give in a single trial a value of the root sufficient for most purposes, and in a few trials a very accurate value.

If \( \sqrt{3} \) be taken \( = \frac{1}{2} \) the first trial gives \( \sqrt{3} = \frac{1}{2} + \frac{1}{2} = 1.7321: \) the true value = 1.73205. A second substitution of \( \frac{1}{4} \) will give the root with great accuracy, the resulting fraction being \( \frac{1}{4} + \frac{1}{4} \approx 1.7320508 \) true to last figure.

Required \( \sqrt[3]{26} \). Suppose 5; then \( \sqrt[3]{26} = \frac{1}{5} + 2\frac{1}{5} = 5.1 \), and a second trial gives 5.0990, which is the root to four places.

Required \( \sqrt[3]{216} = 3\sqrt[3]{24} \). Suppose 5 for \( \sqrt[3]{24} \); then \( \frac{1}{5} + 2\frac{1}{5} = \sqrt[3]{24} = 4.9 \), and 14.7 = \( \sqrt[3]{216} \): the root is 14.6969385. A second trial, using, 14.7 gives 14.696938 true to the last figure.

Required \( \sqrt[3]{37} \). By the formula, supposing the root = 2,

\[
\frac{30}{5 \times 2} + \frac{5}{4} \times 2 = \frac{3}{8} + \frac{8}{5} = 1.975; \text{ the root is 1.97435.}
\]

Required \( \sqrt[3]{37} \). Since \( 3 \sqrt[3]{37} = \sqrt[3]{999} \) and \( (10)^3 = 1000 \),

\[
\frac{999}{3 \times (10)^3} + \frac{2}{3} \times 10 = 3.33 + 6\frac{2}{3} = 9.996666, \text{ and}
\]

\[
9.996666 \times 4 = 33.322222 = \sqrt[3]{37}: \text{ the true root is 3.3222218, being, in a single trial only 4 units in error in the ten millionth place.}
\]
Required \( \sqrt[5]{500000} \). Since \( 2^{10} = 524288 \), suppose \( \sqrt[5]{500000} = 2 \), then

\[
\frac{500000}{19 \times 2^{18}} + \frac{18}{19} \times 2 = \frac{500000}{4980736} + \frac{36}{19} = 1.9950:
\]

The true root seems to be 1.99501.

The advantage of this method appears to be, the ready remembrance of the formula; which for the square root is

\[
\frac{N}{2r} + \frac{r}{2},
\]

and for the cube root

\[
\frac{N}{3r^2} + \frac{2}{3}r;
\]

so that, at least for the square and cube roots, the extraction may be mentally performed. For example, the cube root of 37 has been obtained to six places in a single line of work, which could have been done "in the head;" while to extract it to six figures in the ordinary way requires a good deal of calculation.

It may be noticed that if \( r \) be assumed too great, the first fraction of the formula will be too small, and the second too great; and the contrary if \( r \) be taken too small; so that there is a kind of compensation. If \( r \) be accurately taken, \( R \) will be the exact root; thus, let it be required to find the 10th root of 1024. Suppose the root to be 2, then

\[
\frac{1024}{10 \times 2^9} + \frac{9}{10} \times 2 = \frac{1}{5} + \frac{9}{5} = 2,
\]

which we know to be the correct root.

The fractions resulting for \( R \), when \( r \) is not the exact root, are all in excess; and by assuming different values for \( r \), different series of fractions may be found converging to \( R \). Even if a very erroneous value be assumed for \( r \), in a few trials \( R \) will result with some approach to accuracy; thus if 10 be assumed for \( \sqrt[5]{2} \), four substitutions will give 1.42, omitting the first fractional part until the last substitution.

In the two examples following a little device is exhibited that is often useful.

Required \( \sqrt[5]{91} \). Since \( 2^5 \times 91 = 728 \), and \( 9^5 = 729 \),

\[
\left(\frac{728}{243} + \frac{2}{3.9}\right) + 2 = 4.497942 = \sqrt[5]{91}; \text{ the true root is 4.4979414.}
\]

---

\*That the two members of equation (1) approach equality as the value of \( r \) approximates that of \( R \) may be shown as follows:

Multiply the equation by \( n^{n-1} \) and we have \( N + (n - 1)r^n = nR^{n-1} \)

\( \text{or } N = nR^{n-1} - n^2 + n^4 \).

Assume now \( r = R \) and we have \( N = nR^n - nR^n + R^n \)

\( \text{or } N = R^n. \therefore R = N^\frac{1}{n}. \text{—Ed.} \)
Required $\sqrt[3]{43}$. Since $2^4 \times 43 = 344$; and $7^3 = 343$ we have

\[
\left(\frac{344}{147} + \frac{2}{3} \cdot 7\right) \div 2 = 3.503401; \text{ the true root being } 3503398.
\]

This method is equally applicable to fractional numbers.

Required $\sqrt[2]{5.456789}$. Suppose the root to be $\frac{1}{2}$; then

\[
\frac{5.456789}{2(\frac{1}{2})} + \frac{1}{2} \times \frac{7}{3} = 2.335977; \text{ true to the last figure inclusive.}
\]

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FIRST PRINCIPLES OF THE DIFFERENTIAL CALCULUS.

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BY S. W. SALMON, MOUNT OLIVE, NEW JERSEY.

Let a point $A$ start from $O$ and move uniformly along the line $OX$, and let $dx$ represent its rate of motion. At the same instant that $A$ starts from $O$, let another point $B$ start from $P$ at a distance $b$ from $O$, and move along the line $OU$. Let the distances of the contemporaneous positions of $A$ and $B$ from $O$ be represented by $x$ and $u$ respectively, and let the relation between $x$ and $u$ be expressed by the equation

\[
u = b + ax^n, \ldots \ldots \ldots \ldots (1)
\]

in which $a$ and $b$ are constants. When $n = 1$, $B$ will move uniformly; for any other value of $n$ $B$ moves at a rate which either increases as $x$ increases or decreases as $x$ increases. Our problem is to find a general expression for the rate of motion of $B$, which we will denote by $du$. Let $Q$ and $S$, at distances $u'$ and $x'$ from $O$ respectively, be any two contemporaneous positions of $A$ and $B$. Hence at these points we have

\[
u' = b + ax'^n, \ldots \ldots \ldots \ldots (2)
\]

Let $du'$ be the rate at which $b$ passes through the point $Q$; now when $du'$ is known we can find a point in the line $OU$, $R$ suppose, from which another point $C$ must start simultaneously with $A$ and $B$ so that, moving uniformly along $OU$ at the rate $du'$, it shall coincide with $B$ at the point $Q$. Let $OR = c$. The time in which $B$ passes over the space $u' - b$, being the same as that in which $A$ passes over the space $x' = \text{space} + \text{rate} = x' + dx$. The space which $C$ passes over in this time $= \text{time} \times \text{rate}$

\[
\frac{du'}{dx} x' = u' - C; \therefore c = u' - \frac{du'}{dx} x'. \ldots \ldots \ldots (3)
\]
In the immediate vicinity of $Q$, $C$ will coincide with $B$ only at the point $Q$, for since $C$ moves uniformly at the same rate at which $B$ passes through $Q$, if $B$'s rate is increasing, $B$ will from this point move faster than $C$, and if decreasing slower.

Let $C$ pass over the space $y - c$ while $A$ passes over the space $x$, since $C$ moves uniformly the relation between $x$ and $y$ must be expressed by an equation of the form

$$y - c = cx.$$  \hspace{1cm} (4)

We have evidently

$$\frac{du'}{dx} = \frac{y - c}{x} = \frac{cx}{x} = c.$$  

Substituting this value of $c$, and the value of $c$ from (3) in (4), we get

$$y - u' = \frac{du'}{dx}(x - x').$$  \hspace{1cm} (5)

Subtracting (2) from (1), member from member, we have

$$u - u' = a(x^a - x'^a).$$  \hspace{1cm} (6)

Dividing (6) by (5), member by member, we have

$$\frac{u - u'}{y - u'} = \frac{a(x^a - x'^a)}{x - x'} \frac{du'}{dx};$$

whence

$$\frac{du'}{dx} = a\left(\frac{x^a - x'^a}{x - x'}\right) \left(\frac{y - u'}{u - u'}\right).$$

This equation is true for every corresponding value of $u$, $y$ and $x$; but when

$$x = x', y = u',$$

and

$$\frac{(y - u')}{(u - u')} = \frac{u' - u'}{u' - u'} = 1;$$

whence

$$\frac{du'}{dx} = a\left(\frac{x^a - x'^a}{x - x'}\right) = anx^{a-1}.\text{ }$$

See Davies' Bourdon p. 257. Dropping the accents, since the solution depends on no condition belonging to the particular values of $x$ and $u$, more than to any other corresponding values of $x$ and $u$, we have

$$\frac{du}{dx} = anx^{a-1} \frac{dx}{dx}.$$  

Suppose the relation between $u$ and $x$ to be expressed by the equation

$$u = \varphi(x),$$

where $\varphi(x)$ is a symbol representing any algebraic expression involving $x$.

We have

$$\varphi(x) = \varphi(x' + (x - x')).$$

Suppose the second member of this equation to be developed, and arranged according to the ascending powers of $(x - x')$, then

$$\varphi(x) = \varphi(x') + P(x - x') + Q(x - x')^2 + &c.,$$

where $P$, $Q$, &c., are algebraic expressions containing $x'$. The development must commence with the zero power of $(x - x')$ since it must reduce to $\varphi(x')$ when $x = x'$; whence

$$u - u' = \varphi(x) - \varphi(x') = P(x - x') + Q(x - x')^2 + &c.$$
In the same manner as before we get

$$\frac{du'}{dx} = \frac{y-u'}{u-u'} \left( \frac{\varphi(x) - \varphi(x')}{x-x'} \right) = \frac{y-u'}{u-u'} \left( P + Q(x-x') + \&c. \right);$$

whence, making $x = x'$, we get

$$\frac{du'}{dx} = \left( \frac{\varphi(x) - \varphi(x')}{{x-x'}} \right)_{x=x'} = P.$$

**Note on the Correction of an Error in the Theory of Polyconic Projections, by Prof. W. W. Johnson.**—In the publications of the Coast Survey* it is erroneously stated that in the Polyconic Projection of the Sphere the projected meridians and parallels cut each other at right angles throughout the entire area.

The Polyconic Projection is that in which a central meridian and the equator are represented by straight lines, perpendicular to one another, and correctly subdivided on the same scale; while the parallels are represented by arcs of circles described with radii proportional to the cotangents of the latitudes, such being the radii of the parallels considered as developed each from the tangent cone of which it forms the line of contact with the sphere.

Letting $p$ denote an arc of longitude measured on the equator, $p \cos L$ will denote an arc of the same longitude in latitude $L$, and as in the projection this arc is measured on a circle whose radius is $\cot L$, its circular measure (which is denoted by $\theta$) is $\theta = p \sin L$.

Adopting the central meridian in the projection as axis of $x$, and the equator as axis of $y$, the co-ordinates of the projection of a point in latitude $L$ and longitude $p$ are readily shown to be

$$y = \cot L \sin \theta,$$

$$x = L + \cot L - \cot L \cos \theta.$$

Making $p$ constant these become the co-ordinates of the projected meridian. Now the inclination to the axis of $x$ of a tangent to this curve being denoted by $\varphi$, we shall have

$$\tan \varphi = \frac{dy}{dx};$$

and if the curve cuts the parallel at right angles we shall have the radius of the parallel coinciding with the tangent, and $\varphi$ would be the supplement of $\theta$, or

$$\tan \theta = -\frac{dy}{dx}.$$

*Coast Survey Report for 1853 page 99, "a projection results in which all intersections of parallels and meridians take place at right-angles," and again, "Over the entire area of this projection all parallels and meridians intersect at right-angles." The same error occurs in Church's Descriptive Geometry, Art. 233, on the Polyconic Projection. "This has the advantage that the representatives of the parallels and meridians are perpendicular to each other, as in space, which is not the case in Flamstead's method."
To test the truth of this, we find $d\theta = p \cos L dL$, hence
\[ \frac{dy}{dx} = -\frac{q \cot L \cos \theta \cos L + \csc^2 L \sin \theta}{\cot^2 L + p \cot L \sin \theta \cos L + \csc^2 L \cos \theta}. \]
Putting this expression equal to $\tan \theta$, we deduce
\[ p \cos L \sin^2 \theta - \cot L \sin \theta = -p \cos L \cos^2 \theta, \]
or
\[ p \cos L = \cot L \sin \theta, \]
that is
\[ \sin \theta = p \sin L. \]
But this is not true, since $\theta = p \sin L$; therefore the meridians and parallels do not generally intersect at right angles. The result indicates however that when $\theta$ is small, that is, when either $p$ or $L$ is small, there is a close approximation to perpendicularity. There is also a close approximation when $L$ is near to $90^\circ$.

**Note on the Solution of Prob. 78, by Dr. H. Eggers.**—The published solution of problem 78, short as it is, seems to be rather artificial. The necessity of introducing into the solution the parallelogram equivalent to one-half the triangle does not appear. I beg to present here an other solution, founded on principles of modern geometry.

Let $ANM$ be the given triangle and $MO$ the bisecting line from $M$, if $P$ is situate within the angle $AMN$. From this it is evident that the required bisecting line through $P$ will pass between $O$ and $N$, and between $A$ and $M$. The required line therefore has to cut from the given angle $NAM$ a triangle of equal area with triangle $AOM$. Any other bisecting line, not passing through $P$, will cut on $AN$ and $AM$ respectively two distances $x$ and $y$, reckoning from $A$, so that the rectangle $xy$ is constant and will mark on $AN$ and $AM$ two series of homographic points. Drawing from $P$ lines to every pair of corresponding points we shall have two homographic pencils of rays with the common center $P$. The two coincident rays of the two pencils will be the required bisecting lines. The homography of the two series of points is determined by three pairs of corresponding points, that is, to the points $A$, $O$, $\infty$ on line $AN$ correspond the points $\infty$, $M$, $A$ on line $AM$. Therefore the two pencils are $P(A, O, \infty)$ and $P(\infty, M, A)$. Construct now the two coincident rays according to Steiner's method by means of an arbitrary circle through $P$; these will solve our problem. For the characteristic property of the double ray is, that the two corresponding points which it marks on $AM$ and $AN$, and point $P$ are on one line. —The same solution applies when $P$ is within the triangle, or, at an infinite distance.


§ 1. If we have the two simultaneous linear equations

\[ a_1 x + b_1 y = c_1 \quad a_2 x + b_2 y = c_2 \]

we can, performing the elimination by any of the common methods, obtain values for the variables in terms of the coefficients: thus we find

\[ x = \frac{c_1 b_2 - c_2 b_1}{a_1 b_2 - a_2 b_1} \quad \text{and} \quad y = \frac{a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1} \]

These results may be exhibited more compactly thus

\[ x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \quad \text{and} \quad y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \]

where the symbol \( \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \) means the Determinant of the four quantities enclosed within the vertical lines, that is to say, the algebraic sum of their combinations in sets of two and two, with the condition that each combination contains one, and only one, of the quantities in each row or column: and that the signs are determined by the following rule. The Diagonal of the Determinant, viz., the combination got by reading from the left-hand upper corner to the right-hand lower corner, is affected with the sign plus: and then the sign plus or minus is affixed to each of the remaining combinations, according as it may be derived from the Diagonal by an even or by an odd number of interchanges among the suffixes attached to the quantities \( a_1, b_1 \) &c.

The truth of this rule is rendered evident, in the case before us, by an inspection of the Determinant in its expanded form.

§ 2. Now consider the three equations

\[ a_1 x + b_1 y + c_1 z = d_1 \quad a_2 x + b_2 y + c_2 z = d_2 \quad a_3 x + b_3 y + c_3 z = d_3. \]

From these we obtain, by the common methods of elimination,

\[ x = \frac{d_1 b_2 c_3 - d_1 b_3 c_2 + d_2 b_3 c_1 - d_2 b_1 c_3 + d_3 b_1 c_2 - d_3 b_2 c_1}{a_1 b_2 c_3 - a_1 b_3 c_2 + a_2 b_3 c_1 - a_2 b_1 c_3 + a_3 b_1 c_2 - a_3 b_2 c_1} \]

and similar expressions for the values of \( y \) and \( z \).

The denominator of this value of \( x \) (and also of \( y \) and \( z \)) is a determinant, that is, may be symbolically exhibited as
\[
\begin{vmatrix}
  a_1 & b_1 & c_1 \\
  a_2 & b_2 & c_2 \\
  a_3 & b_3 & c_3 \\
\end{vmatrix}
\]

And the numerator may be written
\[
\begin{vmatrix}
  d_1 & b_1 & c_1 \\
  d_2 & b_2 & c_2 \\
  d_3 & b_3 & c_3 \\
\end{vmatrix}
\]
which may be regarded as having been derived from the Determinant constituting the denominator by erasing the column composed of the coefficients of \(x\) and replacing it by that composed of the absolute terms. And in a precisely similar manner the numerators of the values of \(y\) and \(z\) may be exhibited as Determinants.

§ 3. These illustrations will suffice to give the student a general notion of the nature of Determinants as they occur in the solution of systems of simultaneous linear equations. In the development of the subject however, even in the elementary manner here proposed, it is desirable to give the definitions and establish the properties without direct reference to equations between variables: and this we proceed to do.

§ 4. Definitions. 1°. If there be \(n^2\) quantities arrayed in a square of \(n\) rows and \(n\) columns then the sum with the proper signs (as fixed by the rule given in § 1,) of all possible products of these \(n^2\) quantities in sets of \(n\), one quantity, and only one, being taken from each horizontal row and from each vertical column, is called the Determinant of these \(n^2\) quantities. And the Determinant is said to be of the \(n^{th}\) order.

2°. Each of the \(n^2\) quantities is called an element, and each of the products of \(n\) elements, a constituent of the Determinant. The Determinant is represented by enclosing the \(n^2\) quantities within two vertical lines, as in the examples already given.

3°. If there be \(mn\) quantities arrayed in \(m\) rows and \(n\) columns, and any number of rows and as many columns be selected, the square thus formed is called a Minor of the given array: and if \(n < m\) the minors of the \(n^{th}\) degree are called principal minors. If we desire to represent the Determinants of all the principal minors of the \(mn\) quantities we will enclose the array in double vertical lines;

\[
\begin{vmatrix}
  a_1 & b_1 & c_1 \\
  a_2 & b_2 & c_2 \\
\end{vmatrix}
\]
denotes the three Determinants \[
\begin{vmatrix}
  b_1 & c_1 \\
  b_2 & c_2 \\
\end{vmatrix}, \begin{vmatrix}
  a_1 & c_1 \\
  a_2 & c_2 \\
\end{vmatrix}\text{ and } \begin{vmatrix}
  a_1 & b_1 \\
  a_2 & b_2 \\
\end{vmatrix}.
\]

§ 5. General Properties of Determinants.

1°. The value of a Determinant is not altered if the successive rows are changed into successive columns. Thus

\[
\begin{vmatrix}
  a_1 & b_1 \\
  a_2 & b_2 \\
\end{vmatrix}
= a_1 b_2 - a_2 b_1
= \begin{vmatrix}
  a_1 & a_2 \\
  b_1 & b_2 \\
\end{vmatrix}.
\]

This principle is evident, generally, from the law of formation (§ 1) being perfectly symmetrical with regard to rows and columns.
2°. If any two rows (or columns) be interchanged, the sign of the Determinant will be changed. For this change is evidently equivalent to a single permutation among the suffixes (or letters) and this by the law of formation causes a change of sign.

3°. If two rows (or columns) are identical, the Determinant vanishes. For if we interchange these rows we ought to have a change of sign by the preceding property: but the interchange of two identical rows can produce no change in the value of the Determinant: in other words, the Determinant is equal to itself with its sign changed, but this can only be when it is equal to zero.

4°. If every element in any row (or column) be multiplied by the same factor, the Determinant is multiplied by that factor. This appears from the fact that every constituent of the Determinant contains one and but one element from the same row or the same column.

5°. If the elements in one row (or column) be like multiples of those of another row (or column) the Determinant vanishes. (3° and 4°.)

E. g. \[
\begin{vmatrix} ka_2 & kb_2 & kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.
\]

6°. A Determinant of the \(n\)th order may be expressed in terms of Determinants of the \((n - 1)\)th order. E. g.

\[
\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}
\]

\[
= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}.
\]

When the Determinant is written in this form, the factors multiplying \(a_1\), \(a_2\) &c., or \(a_1, b_1\) &c., are recognised as the minors obtained from the given Determinant by erasing in succession the columns and rows containing \(a_1\), \(a_2\) &c., or \(a_1, b_1\) &c. Let the Minors corresponding to \(a_1, a_2\) &c., be represented by \(A_1, A_2\) &c.: those corresponding to \(b_1, b_2\) &c., by \(B_1, B_2\) &c.: then the original determinant may be briefly written

\[
a_1 A_1 - b_1 B_1 + c_1 C_1 &c.,
\]
or

\[
a_1 A_1 - a_2 A_2 + a_3 A_3 &c.
\]

7°. If all the elements but one of any row (or column) of a Determinant vanish, the order of the Determinant is reduced by one. For if \(a_2, a_3\) &c., in the Determinant last above written all vanish the Determinant becomes \(a_1 A_1\), and \(A_1\) is of the \((n - 1)\)th order.

8°. Hence any Determinant may be exhibited in the form of one of any higher order:
\[ \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ x & a_1 & b_1 \\ y & a_2 & b_2 \end{vmatrix} \]

where \( x, y \) and \( z \) have any values; and this process may be extended indefinitely.

9°. If every element in any row (or column) be a sum, the Determinant is resolvable into the sum of others. Thus, if in the Determinant \( a_1A_1 - b_1B_1 + c_1C_1 \&c., \) we write \( a_1 + k \) for \( a_1, \) \( b_1 + l \) for \( b_1, \) \( c_1 + m \) for \( c_1 \&c., \) the Determinant becomes \((a_1 + k)A_1 - (b_1 + l)B_1 + (c_1 + m)C_1 \&c., \) and this can be written \((a_1A_1 - b_1B_1 + c_1C_1 \&c.,) + (kA_1 - lB_1 + mC_1 \&c.,). \) Similarly, if the elements in any one row (or column) were each the sum of several numbers, the Determinant could be presented as the sum of a like number of Determinants.

10°. If the elements of one row (or column) are respectively equal to those of other rows (or columns) multiplied respectively by constant factors, the Determinant vanishes. For it is the sum of other Determinants which are separately evanescent. E.g.

\[ \begin{vmatrix} kb_1 + lc_1, b_1, c_1 \\ kb_2 + lc_2, b_2, c_2 \\ kb_3 + lc_3, b_3, c_3 \end{vmatrix} = \begin{vmatrix} kb_1, b_1, c_1 \\ kb_2, b_2, c_2 \\ kb_3, b_3, c_3 \end{vmatrix} + \begin{vmatrix} lc_1, b_1, c_1 \\ lc_2, b_2, c_2 \\ lc_3, b_3, c_3 \end{vmatrix} \]

and these latter vanish by 5°.

11°. A Determinant is not altered if we add to each element of any row (or column) the corresponding elements of any other row (or column) multiplied by constant factors. E.g.

\[ \begin{vmatrix} a_1 + kb_1, b_1, c_1 \\ a_2 + kb_2, b_2, c_2 \\ a_3 + kb_3, b_3, c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} kb_1, b_1, c_1 \\ kb_2, b_2, c_2 \\ kb_3, b_3, c_3 \end{vmatrix} \]

but the last Determinant vanishes by 5°.

These principles will now be applied to the calculation of the numerical values of the following Determinants.

§ 6. Examples.

1. \[ \begin{vmatrix} 1 & 1 & 1 \\ 3 & -1 & 2 \\ 4 & 3 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ 3 & -1 \end{vmatrix} - \begin{vmatrix} 3 & 2 \\ 4 & -3 \end{vmatrix} \text{ by } 6°, \]

\[ = (1 - 6) - (-3 - 8) + (9 + 4) = 19. \]

Or thus, using the principle given in 11°; subtract the elements of the first column from the corresponding elements of the second and third columns: this, without altering the value of the Determinant reduces it to

\[ \begin{vmatrix} 1 & 0 & 0 \\ 3 & -4 & -1 \\ 4 & -1 & -5 \end{vmatrix} \text{ which by } 7° \text{ is equivalent to } \begin{vmatrix} -4 & -1 \\ -1 & -5 \end{vmatrix} \text{ or } 20 - 1 = 19. \]
2. \[
\begin{vmatrix}
3 & 4 & -5 \\
4 & -5 & 3 \\
5 & -3 & -4 \\
\end{vmatrix}
= 3 \begin{vmatrix}
-5 & 3 \\
4 & 3 \\
5 & -3 \\
\end{vmatrix}
- 4 \begin{vmatrix}
3 & 3 \\
4 & -3 \\
5 & -3 \\
\end{vmatrix}
- 5 \begin{vmatrix}
4 & 3 \\
4 & -3 \\
5 & -3 \\
\end{vmatrix}
= 3(20 + 9) - 4(-16 - 15) - 5(-12 + 25)
= 146.
\]

3. \[
\begin{vmatrix}
32 & 4 & -5 \\
18 & -5 & 3 \\
2 & -3 & -4 \\
\end{vmatrix}
= 2 \begin{vmatrix}
16 & 4 & -5 \\
9 & -5 & 3 \\
1 & -3 & -4 \\
\end{vmatrix}
= 2(52, 59) = 2(2028 - 1298) = 1460.
\]

The 3rd is obtained from the 2nd by adding to the 2nd and 3rd columns respectively, the elements of the 1st multiplied by 3 and 4 respectively.

4. \[
\begin{vmatrix}
3 & 32 & -5 \\
4 & 18 & 3 \\
5 & 2 & -4 \\
\end{vmatrix}
= 2 \left[ \begin{vmatrix}
9 & 3 \\
1 & -4 \\
\end{vmatrix} + \begin{vmatrix}
16 & 3 \\
5 & -4 \\
\end{vmatrix} + \begin{vmatrix}
4 & 9 \\
5 & 1 \\
\end{vmatrix} \right]
= 2[3(-36 - 3) - 16(-16 - 15) - 5(4 - 45)]
= 2[-117 + 496 + 205] = 2 \times 584 = 1168.
\]

5. \[
\begin{vmatrix}
3 & 4 & 32 \\
4 & -5 & 18 \\
5 & 2 & -3 \\
\end{vmatrix}
= 3 \begin{vmatrix}
-5 & 18 \\
3 & 2 \\
\end{vmatrix}
- 4 \begin{vmatrix}
4 & 18 \\
5 & 2 \\
\end{vmatrix}
+ 32 \begin{vmatrix}
4 & -5 \\
5 & -3 \\
\end{vmatrix}
= 3(-10 + 54) - 4(8 - 90) + 32(-12 + 25)
= 876.
\]

6. (From Salmon's Lessons on Modern Higher Algebra.)

\[
\begin{vmatrix}
9 & 13 & 17 & 4 \\
18 & 28 & 33 & 8 \\
30 & 40 & 54 & 13 \\
24 & 37 & 46 & 11 \\
\end{vmatrix}
= \begin{vmatrix}
1 & 1 & 1 & 1 \\
2 & 4 & 1 & 1 \\
3 & 4 & 1 & 1 \\
4 & 2 & 1 & 1 \\
\end{vmatrix}
= \begin{vmatrix}
1, & 1, & 1, & 1, \\
2, & 4, & 1, & 1, \\
3, & 4, & 1, & 1, \\
4, & 2, & 1, & 1, \\
\end{vmatrix}
= 1, \begin{vmatrix}
1, & 1, & 1, & 1, \\
2, & 4, & 1, & 1, \\
3, & 4, & 1, & 1, \\
4, & 2, & 1, & 1, \\
\end{vmatrix}
= \begin{vmatrix}
1, & 1, & 1, & 1, \\
2, & -1, & -1, & -1, \\
3, & -2, & 2, & -1, \\
4, & 0, & 1, & 1, \\
\end{vmatrix}
= \begin{vmatrix}
4, & -1, & -1, & -1, \\
2, & -1, & -1, & -1, \\
3, & -2, & 2, & -1, \\
4, & 0, & 1, & 1, \\
\end{vmatrix}
= -8 - 7 = -15.
\]

The second Determinant is derived from the first by subtracting from the elements of the first, second and third columns, twice, three times, and four times the corresponding elements of the last column. The remaining steps are very similar.

7. \[
\begin{vmatrix}
a, & b, & c, \\
a, & c, & b, \\
b, & a, & c, \\
\end{vmatrix}
= a \begin{vmatrix}
a, & b, \\
c, & b, \\
\end{vmatrix}
- b \begin{vmatrix}
a, & c, \\
b, & c, \\
\end{vmatrix}
+ c \begin{vmatrix}
a, & a, \\
b, & a, \\
\end{vmatrix}
= a(a^2 - b^2) - b(ab - b^2) + c(c^2 - ab)
= a^3 + b^3 + c^3 - 3abc.
\]

Moreover, since by $11^\circ$ the Determinant is unchanged in value if written

\[
\begin{vmatrix}
a + b + c, & b, & c, \\
a + c + a, & b, & c, \\
b + c + a, & a, & c, \\
\end{vmatrix}
= (a + b + c) \begin{vmatrix}
1, & b, & c, \\
1, & a, & b, \\
1, & c, & a, \\
\end{vmatrix}
\]

it is evident that $a + b + c$ is a factor of $a^3 + b^3 + c^3 - 3abc$. 


8. \[
\begin{vmatrix}
  a & b & c & d \\
  d & a & b & c \\
  c & d & a & b \\
  b & c & d & a \\
\end{vmatrix} = a \begin{vmatrix}
  a & b & c \\
  d & a & b \\
  c & d & a \\
\end{vmatrix} - b \begin{vmatrix}
  d & b & c \\
  c & d & a \\
  b & c & d \\
\end{vmatrix} + c \begin{vmatrix}
  d & a & c \\
  c & d & b \\
  b & c & a \\
\end{vmatrix} - d \begin{vmatrix}
  d & a & b \\
  c & d & a \\
  b & c & d \\
\end{vmatrix} = \left\{ \begin{array}{l}
  a^4 - b^4 + c^4 - d^4 - 2a^2b^2 \\
  + 2b^2c^2 - 4a^2bd + 4b^2ac \\
  - 4c^2bd + 4d^2ac,
\end{array} \right.
\]

and as in Ex. 7, it appears that \(a + b + c + d\) is a factor of this Determinant.

9. \[
\begin{vmatrix}
  0 & c & b \\
  c & 0 & a \\
  b & a & 0 \\
\end{vmatrix} = -c(0 - ab) + b(ac - 0) = 2abc.
\]

§ 7. Applications of Determinants.

If we write the Determinant

\[
\begin{vmatrix}
  a_1 & b_1 & \ldots & n_1 \\
  a_2 & b_2 & \ldots & n_2 \\
  \vdots & \vdots & \ddots & \vdots \\
  a_n & b_n & \ldots & n_n \\
\end{vmatrix}
\]

in the form given in 6°, § 5, viz., \(a_1 A_1 - a_2 A_2 + a_3 A_3 \ldots \pm a_n A_n\) it is easy to establish the following relations:

\[
b_1 A_1 - b_2 A_2 \ldots \pm b_n A_n = 0,
\]

\[
c_1 A_1 - c_2 A_2 \ldots \pm c_n A_n = 0, \text{ &c., for the left-hand members are evidently what the Determinant becomes when } b_1, b_2, \text{ &c., or } c_1, c_2, \text{ &c., are written for } a_1, a_2, \text{ &c., that is to say, when there are two identical rows: but by } 3°, \text{ § 5, in this case the Determinant vanishes.}
\]

§ 8. Let there be given the following system of simultaneous linear equations,

\[
a_1 x + b_1 y + c_1 z = k_1 \\
a_2 x + b_2 y + c_2 z = k_2 \\
\ldots \ldots \ldots \ldots \ldots = k_n
\]

and let the first be multiplied by \(A_1\), the second by \(-A_2\), the third by \(A_3\) &c., and all added: the coefficient of \(x\) will be \(a_1 A_1 - a_2 A_2 \text{ &c.}, \) and those of the other variables will vanish by virtue of the relations established in the preceding article. Thus we have

\[
(a_1 A_1 - a_2 A_2 \text{ &c.}) x = k_1 A_1 - k_2 A_2 + k_3 A_3 \text{ &c.}
\]

Here we observe that the coefficient of \(x\) is the Determinant of the square formed of the coefficients of the variables, and the right-hand member is evidently the Determinant of the square got by erasing the column of the coefficients of \(x\) and replacing it by that composed of the absolute terms. Let us denote the former by \(V\) and the latter by \(D_1\). Then the value of \(x\) as above determined is \(x = \frac{D_1}{V}\).
Returning to the given equations, and multiplying them by $B_1$, $B_2$, $B_3$ &c., and adding, the coefficients of all the variables except $y$ will vanish, while that of $y$ will be $b_1B_1 - b_2B_2$ &c., which by § 5 is equivalent to the Determinant $V$; and the sum of the right-hand members will be $k_1B_1 - k_2B_2 + k_3B_3$ &c., and this is the Determinant of the square got by erasing the coefficients of $y$ and replacing them by the corresponding absolute terms: calling this Determinant $D_2$, we have $V \cdot y = D_2$;

\[ y = \frac{D_2}{V^2}. \] Similarly, $z = \frac{D_3}{V}$ and so for the other variables.

**Example.** Given \[
\begin{align*}
3x + 4y - 5z &= 32 \\
4x - 5y + 3z &= 18 \\
5x - 3y - 4z &= 2
\end{align*}
\] to find $x$, $y$, and $z$.

The Determinants $V$, $D_1$, $D_2$, and $D_3$ are those whose values have already been computed in Examples 2, 3, 4 and 5 in § 6. Referring to them we have \[
x = \frac{D_1}{V} = \frac{1460}{146} = 10; \quad y = \frac{D_2}{V} = \frac{1168}{146} = 8, \quad \text{and} \quad z = \frac{D_3}{V} = \frac{876}{146} = 6.
\]

§ 9. The following results follow as corollaries from § 8.

1°. If $V$ vanishes the values of the variables become infinite, thus indicating that the proposed equations are inconsistent.

2°. If $V = 0$ and the absolute terms also all vanish, the values of the variables as furnished by the method of § 8 assume the form $0 \div 0$. It is practicable however to determine the ratios of $n - 1$ of the unknown quantities to the remaining unknown, thus:—take the three equations

\[ a_1x + b_1y + c_1z = 0, \quad a_2x + b_2y + c_2z = 0, \quad a_3x + b_3y + c_3z = 0; \]

we may write them in the form

\[ a_1 \frac{x}{z} + b_1 \frac{y}{z} = -c_1, \quad a_2 \frac{x}{z} + b_2 \frac{y}{z} = -c_2, \quad a_3 \frac{x}{z} + b_3 \frac{y}{z} = -c_3; \]

solving these for the ratios $\frac{x}{z}$ and $\frac{y}{z}$ we have, (by § 8), using the first two equations, the second and third equations, and the first and third equations, respectively:

\[
\frac{x}{z} = \begin{vmatrix} -c_1 & b_1 \\ -c_2 & b_2 \\ a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} -c_1 & b_1 \\ -c_3 & b_3 \\ a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} = \begin{vmatrix} -c_1 & b_1 \\ -c_3 & b_3 \\ a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.
\]

And similarly $\frac{y}{z}$,

\[
\frac{y}{z} = \begin{vmatrix} a_1 & -c_1 \\ a_2 & -c_2 \\ a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & -c_1 \\ a_3 & -c_3 \\ a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 & -c_1 \\ a_3 & -c_3 \\ a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.
\]
These three forms of solution can be shown to coincide; and the same process may be extended to $n$ equations involving $n$ unknown quantities.

3°. If the absolute terms all vanish and $V$ does not vanish, the only possible solutions are $x = 0$, $y = 0$ &c. Thus it appears that in order that a system of $n$ homogeneous equations in $n$ variables shall have values for the variables other than zero, it is necessary that the Determinant $V$ shall vanish.

4°. If we have $n$ non-homogeneous equations containing $n - 1$ variables $V = 0$ is a necessary condition of consistence and independence; and by combining any $n - 1$ of the equations (as in 2°) the values of the $n - 1$ variables are found.

§ 10. To find the relation which must hold among the coefficients $A$, $B$ &c. in order that the quadratic function $Ax^2 + By^2 + Cz^2 + 2Dxy + 2Eyz + 2Fzx$ (1) shall break into two factors of the first degree in $x$, $y$ and $z$.

The proposed function may be written $(Ax + Dy + Fz)x + (By + Dx + Ez)y + (Cx + Ey + Fz)z$ (2). If the relation between the coefficients is such that, for all values of $x$, $y$ and $z$, the linear functions within the parenthesis have some constant ratio, say $p: q: r$, we have $By + Dx + Ez = \frac{q}{p} (Ax + Dy + Fz)$, and $Cz + Ey + Fz = \frac{r}{p} (Ax + Dy + Fz)$; then (2) may be written $(x + \frac{q}{p} y + \frac{r}{p} z) \times (Ax + Dy + Fz)$. This therefore is the condition to be fulfilled: it may be written

$$\frac{Ax + Dy + Fz}{p} = \frac{By + Dx + Ez}{q} = \frac{Cz + Ey + Fz}{r};$$

and since these relations are to hold regardless of the values of $x$, $y$ and $z$, they will hold for these, which reduce the numerators to zero: that is, we have

$$Ax + Dy + Fz = 0,$$
$$By + Dx + Ez = 0,$$
$$Cz + Ey + Fz = 0.$$

Eliminating the variables we get

$$\begin{vmatrix} A & D & F \\ D & B & E \\ F & E & C \end{vmatrix} = 0$$

as the necessary relation between the coefficients in order that the function proposed shall break into two factors. This Determinant is, by some writers, called the Discriminant of the proposed function.

This is all that our limits allow on this subject. Students desiring to pursue the subject are recommended to read Baltzer’s Treatise on Determinants, Salmon’s Lessons on Modern Higher Algebra, Spottiswoode, Todhunter, etc.
DIVISIBILITY BY PRIME NUMBERS.

BY PROF. D. M. SENSENIG, NORMAL SCHOOL, INDIANA, PA.

Theorem. A prime number \( p \), other than 2 and 5, is a divisor of a number \( N \) if it is a divisor of the sum of its digits, taken in groups of as many figures each, as there are figures in the repetend produced from the fraction \( 1 \div p \).

Illustration. Thus 37 is a divisor of 61210122865, since it is a divisor of 865 + 122 + 210 + 61, or 1258. Notice that \( 1 \div 37 \) yields a repetend of three figures.

Demonstration. Put \( r \) for the number of figures in the repetend produced from \( 1 \div p \), \( t \) for the number of groups of \( r \) figures each in the number, and \( n_1, n_2, n_3, \ldots, n_{t-1}, n_t \), for the number of units in the successive periods, beginning at the right; then

\[
N = n_t(10)^r + n_{t-1}(10)^{r-1} + \ldots + n_3(10)^3 + n_2(10)^2 + n_1(10).
\]

But from the nature of repetends, \( p \) divides each of the terms \( (10)^r, (10)^{r-1}, \ldots, (10)^3, (10)^2, \) and \( (10) \), with a remainder of 1, hence it divides \( N \) with a remainder of \( n_t + n_{t-1} + \ldots + n_3 + n_2 + n_1 = R \). Now if \( R \) is divisible by \( p \) it is evident that \( p \) is a divisor of \( N \); but \( R \) is the sum of the digits taken in groups of \( r \) figures together, hence \( p \) is a divisor of \( N \), if it is a divisor of the sum of its digits taken in groups of \( r \) figures together.

Corollaries. 1. Three is a divisor of a number if it is a divisor of the sum of its digits taken singly, since \( \frac{1}{3} \) produces a repetend of one figure.

2. Eleven is a divisor of a number if it is a divisor of the sum of its digits taken two together, since \( \frac{1}{11} \) produces a repetend of two figures.

3. Thirty seven is a divisor of a number if it is a divisor of the sum of its digits taken three together, since \( \frac{1}{37} \) produces a repetend of three figures.

4. One hundred and one is a divisor of a number if it is a divisor of the sum of its digits taken four together, since \( \frac{1}{101} \) produces a repetend of four figures.

5. Forty one and two hundred and seventy one are divisors of a number if they are divisors of the sum of its digits taken five together, since \( \frac{1}{41} \) and \( \frac{1}{271} \) each produces a repetend of five figures.

6. Seven and thirteen are divisors of a number if they are divisors of the sum of its digits taken six together, since \( \frac{1}{7} \) and \( \frac{1}{13} \) each produces a repetend of six figures.

7. Any prime number that produces a perfect repetend is a divisor of a number if it is a divisor of the sum of its digits taken as many figures together as there are units in the prime number less one.
8. Since \((\frac{999\ldots n}{9})\) is the general form of a common fraction producing a
repetend of \(r\) figures, it follows that any divisor of \((999\ldots 9)\), is a divisor
of a number if it is a divisor of the sum of its digits taken \(r\) together.

NOTE. In a letter to the Editor, Prof. H. T. Eddy writes:—"In the
very interesting Historical Sketch contained in your Sept. No. there is one
omission which I feel should be supplied in the enumeration of the articles
contributed to the Mathematical Monthly. I refer to Ferrel's investigations
respecting the laws which govern atmospheric currents. These articles are
regarded, I think, as the most important original investigations published
in the Mathematical Monthly."

SOLUTIONS OF PROBLEMS IN NUMBER SIX, VOL. II.

Solutions of problems in No. 6, Vol. II, have been received as follows:
From J. M. Arnold, 92; Prof. W. W. Beman, 93 and 94; Lient. S. H.
Baker U. S. N., 94; G. L. Dake, 92 and 95; G. M. Day, 97; Cadet E. S.
Farrow, 92, 93, 94, 95 and 97; Henry Gunder, 92, 95 and 97; Christine
Ladd, 93, 94 and 95; Artemas Martin, 92 and 95; Dr. A. B. Nelson, 92
and 95; O. D. Oathout, 92, and 97; Geo. H. Pegram, 95; K. M. Supten,
92 and 95; Prof. J. Scheffer, 92, 93, 94, 95 and 97; E. B. Seitz, 95 and 97;
E. H. Westermann, 95; Prof. C. M. Woodward, 97.

92.—"A balloon is ascending vertically with a given velocity \(v\), and a
body is let fall from it, which touches the ground in \(t\) seconds; find the
height of the balloon at the moment the body is let fall from it."

SOLUTION BY HENRY GUNDER, NORTH MANCHESTER, IND.

In \(t\) seconds a body will fall from rest \(\frac{1}{2}gt^2\) feet. But from the conditions
of the problem it ascends \(vt\) feet. Therefore it falls from a height of \(\left(\frac{1}{2}gt-v\right)t\)
feet.

93.—"To construct a triangle if the three radii of the circles, which touch
the three sides externally, are given."

SOLUTION FURNISHED BY PROF. W. W. BEMAN, ANN ARBOR, MICH.

Christine Ladd writes: "The construction follows at once from the solu-
tion given by Chauvenet, 164, 12." Prof. Beman writes: "The following
elegant solution of Problem 93 may be found — in substance — in Catalan's
'Theoremes et Problemes de Geometrie Elementaire', page 155:
Let \( r, r', \) and \( r'' \) be the radii of the escribed circles, \( a, b, \) and \( c, \) the corresponding sides of the triangle.

Equating three expressions for twice the area of the triangle, we have
\[
\frac{b+c-a}{r'} = \frac{c+a-b}{r} = \frac{a+b-c}{r''-r'},
\]
whence we easily obtain
\[
\frac{a}{r + rr' + r''} = \frac{b}{r' + rr' + r''} = \frac{c}{r + r'}.
\]
The triangle required is therefore similar to the one whose sides are \( r + rr' + r'', \) \( r' + rr' + r'' \), and \( r + r' \).

Construct this triangle, draw its escribed circles, and through points at distances from their centres equal to \( r, r', \) and \( r'' \), draw parallels to its sides. We then have the triangle.”

94.—“One side of a quadrilateral, whose four sides are given in length, is fixed: find the equation of the locus of the middle point of the opposite side, in rectangular coordinates.”

**SOLUTION BY CHRISTINE LADD, UNION SPRINGS, N. Y.**

We have the following equations from which to determine the relation between \( x \) and \( y, \) if the origin is at one extremity of the fixed side.
\[
x = \frac{1}{2}(x' + x''), \quad y = \frac{1}{2}(y' + y''),
\]
\[
x'^2 + y'^2 = a^2, \quad (x'' - a)^2 + y''^2 = b^2, \quad (x' - x'')^2 + (y' - y'')^2 = c^2.
\]

On elimination an equation of the sixth degree is obtained. When the quadrilateral is a rhombus and the origin is at the middle point of the fixed side, whose length is \( 2a, \) the equation reduces to
\[
x^6 + 3x^4y^2 + 3x^2y^4 - 8ax^4 - 12ax^2y^2 + 16a^2x^2 - 4a^2y^2 + y^4 = 0.
\]
This is the equation of three circles with radii respectively equal to \( a, a \) 2\( a, \) and centres at \( (a, 0) \) \(-a, 0)\) and \((0, 0)\).

**SOLUTION BY PROF. J. SCHEFFER, COLLEGE OF ST. JAMES, MD.**

By referring to the diagram the following equations are easily verified:
\[
\begin{align*}
b \sin \alpha + d \sin \beta &= 2y, \quad \ldots \quad (1) \\
b \cos \alpha + \frac{1}{2}c \cos \gamma &= x, \quad \ldots \quad (2) \\
d \cos \beta + \frac{1}{2}c \cos \gamma &= a - x, \quad \ldots \quad (3) \\
b \sin \alpha - d \sin \beta &= c \sin \gamma, \quad \ldots \quad (4)
\end{align*}
\]
By combining (1) and (4), and (2) and (3) we get
\[
\sin \alpha = \frac{2y + c \sin \gamma}{2b}, \quad \ldots \quad (5)
\]
\[
\sin \beta = \frac{2y - c \sin \gamma}{2d} \quad (6), \ \cos \alpha = \frac{2x - c \cos \gamma}{2b} \quad (7), \ \cos \beta = \frac{2(a-x) - c \cos \gamma}{2d} \quad (8)
\]

Squaring and adding (5) and (7), and also (6) and (8) we have,
\[
8y^2 + 2c^2 + 4x^2 + 4(a - x)^2 - 4ac \cos \gamma = 4b^2 + 4d^2,
\]
whence
\[
\cos \gamma = \frac{4y^2 + 4x^2 - 4ax + c^2 - 4b^2 - 4d^2}{2ac}.
\]

Substituting this value of \( \cos \gamma \) in (7) and (8) we obtain
\[
\cos \alpha = \frac{8ax - 4y^2 - 4x^2 - c^2 + 4b^2 + 4d^2}{4ab},
\]
\[
\cos \beta = \frac{4a^2 - 4y^2 - 4x^2 - c^2 + 4b^2 + 4d^2}{4ad}.
\]

Adding (2) and (3) we have \( b \cos \alpha + d \cos \beta + c \cos \gamma = a. \)

Substituting in this equation for \( \cos \alpha, \cos \beta \) and \( \cos \gamma \), their values as found above we obtain for the equation of the locus,
\[
4y^2 + 4x^2 - 4ax = 4b^2 + 4d^2 - 2a^2 - c^2,
\]
or
\[
y^2 + (x - \frac{1}{2}a)^2 = b^2 + d^2 - \frac{1}{4}(a^2 + c^2).
\]

Consequently the locus is a circle, whose centre is the middle point of the side \( AB \), and whose radius is \( \sqrt{b^2 + d^2 - \frac{1}{4}(a^2 + c^2)} \).

[Prof. Beman has given the equation of the locus of any point in the line \( DC \). His equation is of the 6th degree. Prof. Woodward writes: "Prob. 94, is a special case of the well-known general problem connected with Watt's parallel motion. The full solution, with careful discussion is given by Prony in his \textit{Architecture Hydraulique}."]

95.—"Prove, otherwise than by the Integral Calculus, that
\[
\frac{\pi}{2} - \sin^{-1} e = 2 \tan^{-1} \left( \frac{1 - e}{1 + e} \right) ^{\frac{1}{2}}.
\]

\textbf{Solution by Dr. Nelson.}—Let \( e = DB \)

\( = \sin EB \). Then \( AB = \frac{1}{2} \pi - \sin^{-1} e \).

Make \( AF = DB \) and complete the rectangle \( CFGD \). Then, radius being unity,

\( CF = 1 + e, GF = (1 - e)^{\frac{1}{2}}, \) and \( \tan OA \)

\( = \frac{(1-e^{\frac{1}{2}})}{1+e} = \left( \frac{1-e}{1+e} \right) ^{\frac{1}{2}} \). We have to show

\( AB = 2OA \). Draw \( CB \) and \( AG \).

\( CAGB \) is easily seen to be a rhombus; hence

\( CG \) bisects \( A CB, \) & \( AB = 2AO. \)

96.—"\( A \) plays \( m \) games with \( B \) whose skill is equal to his own. Required the probability that one of them will win \( n \) consecutive games."
[Several solutions of this question have been received, but in all of them
the probability of winning \( n \) consecutive games merely is considered, and no
account is taken of the fact that there are \( m \) games in which the run may
occur. If \( m \) is greater than \( n \), as is presumably the case in this question,
the chance of a run of \( n \) games is obviously greater than if \( n \) games only
were played.

When \( m \) is a small number it is not difficult to determine the required
chance, as all the possible results may be presented to the eye, and the
runs of \( n \) games counted, thus:

If \( m = 6 \) and \( n = 2 \), the number of possible different events is \( 2^6 = 64 \),
by writing down which, and counting, we find, in the 64 sets of 8 games
each which represent all the possible events, 43 in which a run of two or
more occurs. Hence the chance that either one of the players will win two
consecutive games in a set of six games is \( \frac{43}{64} \), or more than one-half.

When \( m \) is a large number, however, this method becomes impracticable,
and the solution involves a somewhat intricate analysis. De Morgan has
given the solution of an analogous question as an example of the application
of Laplace's Theory of Generating Functions. He gets, as a representative
of the required probability, the following formula:

\[
p^n [p'(m - n) + 1] - p^n p' \frac{m - 2n}{1 \cdot 2} \left[ p'(m - 2n - 1) + 2 \right]
+ p^n p'' \frac{(m - 3n - 1)(m - 3n)}{1 \cdot 2 \cdot 3} \left[ p'(m - 3n - 2) + 3 \right] \quad \text{etc}
\]

In this formula \( p \) is the probability of a single event, \( \frac{1}{2} \) in this problem,
and \( p' = 1 - p. \quad \text{Ed.} \]

97.—“Find the average distance of all the points of a sphere, radius \( r \),
from a point whose distance from the center is \( a \).”

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

There are two cases. The given point may be within the sphere, or
without the sphere.

1. Let \( A \) be the given point, \( O \) the center, and \( P \)
any point of the sphere. Draw \( PM \) perpendicular to
\( AO \).

Put \( OB = r, OA = a, OM = x, PM = y \). Then
\( AP = [(a - x)^2 + y^2]^\frac{1}{2}, \) when \( x < a, \) and \( AP \)
\( = [(a - x)^2 + y^2]^\frac{1}{2}, \) when \( x > a; \) hence the sum of
the distances of all the points of the sphere from \( A \) is
\[ \int \int_0^2 \pi y ((a - x)^2 + y^2) y \, dx \, dy + \int \int_0^2 \pi y ((x - a)^2 + y^2) y \, dx \, dy \]

(in which the superior limit \( u = \sqrt{r^2 - x^2} \))

\[ = \frac{3}{\pi} \int [r^2 - 2ax] \, dx + \frac{3}{\pi} \int [(a^2 + r^2 - 2ax)^{\frac{r}{a}} (r - a)^{\frac{r}{a}} - \frac{1}{r^2} (r + a)^{r/2} + (r - a)^{r/2}] \]

\[ = \frac{3}{5a} [(r + a)^{r/2} - (r - a)^{r/2}] - \frac{1}{r^2} [(r + a)^{r/2} + (r - a)^{r/2}] \]

\[ = \frac{3}{5a \pi^2} (3r^2 + \frac{a^2}{2r} - \frac{a^4}{20r^3}) \]

Hence the average distance is

\[ \frac{3}{5a \pi^2} \left( \frac{3r^2}{4} + \frac{a^2}{2r} - \frac{a^4}{20r^3} \right) + \frac{3}{\pi^2} = \frac{3}{4} + \frac{a^2}{2r} - \frac{a^4}{20r^3} \]

2. When the given point is without the sphere, the sum of the distances of all the points of the sphere from \( A \) is

\[ \int \int_0^2 \pi y ((a - x)^2 + y^2) y \, dx \, dy = \frac{3}{\pi} \int [r^2 - 2ax] \, dx + \frac{3}{\pi} \int [(a^2 + r^2 - 2ax)^{\frac{r}{a}} (r - a)^{\frac{r}{a}} - \frac{1}{r^2} (r + a)^{r/2} + (r - a)^{r/2}] \]

\[ = \frac{3}{5a} [(r + a)^{r/2} - (r - a)^{r/2}] - \frac{1}{r^2} [(r + a)^{r/2} + (r - a)^{r/2}] \]

Hence the average distance is \( \frac{3}{5a \pi^2} (a + \frac{r^2}{5a}) + \frac{3}{\pi^2} = a + \frac{r^2}{5a} \). When \( a = r \), the above results both reduce to \( \frac{3}{5r} \).

The "QUERY" in No. 6, Vol. II, was answered by E. B. Seitz, E. S. Farrow, Prof. J. Scheffer, Prof. A. B. Evans, Prof. A. Hall, and by Prof. H. T. Eddy. All the solutions are elegant, and most of them brief. The following is by Prof. Eddy of Cincinnati, Ohio.

\[ u = \int_0^\pi \frac{x \sin x \, dx}{(1 - p^2 \sin^2 x)^{\frac{1}{2}}} \]

By parts,

\[ u = \frac{1}{1 - p^2} \left[ \frac{x \cos x}{(1 - p^2 \sin^2 x)^{\frac{1}{2}}} - \int (1 - p^2 \sin^2 x)^{\frac{1}{2}} \cos x \, dx \right]_0^\pi \]

\[ = \frac{1}{1 - p^2} \left[ \frac{x \cos x}{(1 - p^2 \sin^2 x)^{\frac{1}{2}}} - \frac{1}{p} \sin^{-1} (p \sin x) \right]_0^\pi \]

\[ = \frac{\sin^{-1} p}{p (1 - p^2)} \]

[After the foregoing solutions were put in type we received, from J. M. Greenwood and W. H. Baker, solutions of 92, 93, 95, 96 and 97. The following is the formula they obtain for the solution of 96:

\[ \left( \frac{1}{2} \right)^m (r + 1 + \frac{r(r - 1)^2}{1 \cdot 2} + \frac{r(r - 1)(r - 2)^2}{1 \cdot 2 \cdot 3} + \ldots + \frac{r}{r} \]

In this formula \( m = n = r \), and \( n > r \).]
PROBLEMS.

98. **By Cadet E. S. Farrow, West Point, N. Y.**—A stream of water moves at the rate of 9 miles an hour, with a fall of one foot per mile; what is the momentum of the water per square foot?

99. **By Prof. J. Scheffer, College of St. James.**—Two points are given. Without using a ruler—that is with a pair of dividers only—to determine two other points which, with the first two, form the four vertices of a square.

100. **By Prof. A. B. Evans, Lockport, N. Y.**—Three circles \( A, B, C \) are tangent externally, and within the space enclosed by this group, three other circles \( A_1, B_1, C_1 \) are inscribed tangent to one another and each tangent to two circles of the first group: the group \( A_x, B_x, C_x \) is similarly situated with respect to the group preceding it. If \( a, b, c \) and \( a_x, b_x, c_x \) be the radii of the circles in the first and \((x+1)\)th groups' prove the following relations;

\[
\frac{1}{a_x} + \frac{1}{a} = \frac{1}{b_x} + \frac{1}{b} = \frac{1}{c_x} + \frac{1}{c} \quad \text{when } x \text{ is odd};
\]

and

\[
\frac{1}{a_x} - \frac{1}{a} = \frac{1}{b_x} - \frac{1}{b} = \frac{1}{c_x} - \frac{1}{c} \quad \text{when } x \text{ is even}.
\]

101. **By Prof. C. M. Woodward, St. Louis, Mo.**—From any point \( O \), within the circumference of a circle, two lines are drawn making a constant angle with each other. These lines revolve about \( O \) in the plane of the circle, and from the points where they cut the circumference tangents are drawn. Find the locus of the intersection of these tangents.

102. **By G. W. Hill.**—Prove that in every triangle the square of the sum of the squares of the sides exceeds double the sum of their fourth powers.

103. Through two given points draw a circle bisecting the circumference of a given circle.

104. **By Dr. A. B. Nelson, Danville, Ky.**—Show that \( x^2 + 1 = y^2 \) is possible for the values \( x = 0, -1, \) and \( 1, \) only.

105. Show that in taking a handful of shot from a bag it is more probable that an odd number will be drawn than an even number.

106. **By Christine Ladd, Union Springs, N. Y.**—Interpret

\[
\frac{\alpha \beta^2 - \beta \alpha^2}{V\alpha \beta}.
\]
BOOK NOTICES.


On the Transcendental Curves whose Equation is \( \sin y \sin m y = a \sin x \sin nx + b \). By H. A. Newton and A. W. Phillips.

This is an octavo pamphlet containing 11 pages of discussion of the properties of the various curves which result from the substitution of different values for the four arbitrary constants \( a, b, m, n \); and 23 pages of plates. These plates are very elaborate, and are executed in superior style.


We take pleasure in calling the attention of students of this interesting branch of analysis, to this extremely well executed and comprehensive text-book upon the subject. From the limited perusal we have been able to give this book we feel justified in saying, that the author has treated the various parts of his subject in a clear, concise, and comprehensive manner; and we think the student will encounter no greater difficulties in following the author in his discussions, than is usually encountered by students of Algebra in mastering that branch of analysis.

EDITORIAL REMARKS.

From the kind letters we receive offering pecuniary aid, in case the receipts for the Analyst are not sufficient to keep it going, we are induced to say to our subscribers, in commencing the third volume of the Analyst, that we are by no means discouraged in the enterprise; on the contrary, the patronage we have received, considering the various disadvantages under which our publication appears, is better than we had reason to hope for; and we can account for it only on the assumption that mathematicians are disposed to support a mathematical publication, wherever, and by whomsoever, it may be inaugurated.

Hence, though it does not yet pay us, in a pecuniary point of view, we have no thought of abandoning the enterprise. We have no doubt but that, if our health permits, we shall be able, with a few more years of experience in its management, to place the Analyst in a condition that will enable us, when we are compelled to retire, to transfer its management to abler hands and to a more congenial locality.

We tender our sincere thanks to those who have offered to assist us, and reply, that all we ask is, that the friends of the publication will make such reasonable effort to extend its circulation among mathematicians and students of mathematics as may be thought consistent.

ERRATA.

On page 9, line 13, for "\( x^n \)" in numerator of last term, read \( n^x \).

11, " 5 from bottom, for "\( \frac{1}{x} \)" read \( \frac{1}{\pi} \).

13, " 5 dele "inclusive."
NEW DEMONSTRATION AND FORMS OF LAGRANGE'S THEOREM. THE GENERAL THEOREM.

BY LEVI W. MEECH A. M. HARTFORD CONN.

"In the case of Lagrange's Theorem, Lambert of Alsace (died 1777), in endeavoring to express the roots of Algebraic equations in series, found a law resembling that which we have just developed. He published his results in 1758, and Lagrange generalized them into the theorem, which bears his name, in 1772. Finally, in the Mecanique Celeste, Laplace made a still further extension." (DeMorgan's Calculus, p. 171).

The following investigation from a new point of view, appearing entirely conclusive, is presented to facilitate the study of analysis. At the close, the reasoning unexpectedly leads to the general Theorem, under a simple exponential form, which includes this whole class of Theorems, especially those of Taylor, Lagrange and Laplace, as particular cases, according to the initial differentiations there described.

To find the value of \( x \) in terms of \( t \) and \( e \) from the primary equation,

\[
x = t + efz = t + h.
\]

Let us make the single letter \( h = efz \), as already denoted; then taking the same function \( f \) of each side,

\[
fz = f(t + h).
\]

The first member \( fx = h + e \), as indicated above. And this equals the second member developed in powers of \( h \) by Taylor's Theorem:

\[
\frac{h}{e} = ft + \frac{df}{dt} \cdot h + \frac{d^2 ft}{dt^2} \cdot \frac{h^2}{1 \cdot 2} + \frac{d^3 ft}{dt^3} \cdot \frac{h^3}{1 \cdot 2 \cdot 3} + \ldots
\]

The auxiliary \( h \) can now be found by simple reversion of series; thus let,

\[
h = Ae + Be^3 + Ce^5 + De^7 + \ldots .
\]
Substituting this series in place of \( h \) in equation (3) and equating the coefficients of equal powers of \( e \); also denoting \( ft \) by the single letter \( f \),

\[
A = ft = f,
\]

\[
B = \frac{df}{dt} \cdot A = \frac{1}{2} \frac{d^2 f}{dt^2} f^2,
\]

\[
C = \frac{df}{dt} \cdot B + \frac{1}{2} \frac{d^2 f}{dt^2} \frac{d^2}{dt^2} A^2 = \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \frac{d^4 f}{dt^4} f^2,
\]

\[
D = \frac{df}{dt} \cdot C + \frac{1}{2} \frac{d^2 f}{dt^2} \frac{d^2}{dt^2} A B + \frac{1}{2} \frac{d^2 f}{dt^2} \frac{d^2}{dt^2} A^2 = \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \frac{d^4 f}{dt^4} f^2 + \ldots
\]

\( h = ef + \frac{1}{1 \cdot 2} \frac{d^2 f^2}{dt^2} + \frac{1}{1 \cdot 2 \cdot 3} \frac{d^3 f^2}{dt^3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \frac{d^4 f^2}{dt^4} + \ldots \)

The regular law of the series continues to any extent, as will be shown presently. Substituting in equation (1) we have a particular case of Lagrange's Theorem:

\[
x = t + ef + \frac{1}{1 \cdot 2} \frac{d}{dt} (ef^2) + \ldots + \frac{1}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n} \frac{d^{n-1}}{dt^{n-1}} (ef^n) + \ldots
\]

Generally, taking any function of each side of equation (1), and developing by Taylor's Theorem,

\[
F_x = F(t + h) = Ft + \frac{dF}{dt} \cdot h + \frac{d^2 F}{dt^2} \cdot \frac{h^2}{1 \cdot 2} + \ldots
\]

Here, substituting for \( h \) its value from equation (5), and reducing, we have Lagrange's Theorem:

\[
F_x = Ft + ef \cdot \frac{dF}{dt} + \frac{1}{1 \cdot 2} \frac{d}{dt} (ef^2 \cdot \frac{dF}{dt}) + \frac{1}{1 \cdot 2 \cdot 3} \frac{d^2}{dt^2} (ef^3 \cdot \frac{dF}{dt}) + \ldots
\]

\[
+ \frac{1}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n} \frac{d^{n-1}}{dt^{n-1}} (ef^n \cdot \frac{dF}{dt}) + \ldots
\]

It only remains to prove that the regular law of the first four terms of the series, continues uniform. Developing each member of the last equation by Maclaurin's Theorem, and equating coefficients of equal powers of \( e \),

\[
\left[ \frac{d^2 F_x}{dt^2} \right] = \frac{d}{dt} \left( \frac{d}{dt} \cdot \frac{dF}{dt} \right); \quad \left[ \frac{d^3 F_x}{dt^3} \right] = \frac{d^2}{dt^2} \left( \frac{d}{dt} \cdot \frac{dF}{dt} \right).
\]

The simple law of third Maclaurin derivation by increasing the exponent of derivation and of \( ft \), each by unity, is obvious. And since it is applicable to any form of the function \( F \), let a new function \( F' \) be assumed, such that,

\[
\left[ \frac{d^3 F_x}{dt^3} \right] = \frac{d^2}{dt^2} \left( \frac{d}{dt} \cdot \frac{dF}{dt} \right) = \left[ \frac{d^3 F_x}{dt^3} \right].
\]
Differentiating these equals successively, we see that the law of the advance in $Fx$ must coincide with the law of the rear terms already established in both $Fx$ and $Ft$; and so on to infinity. Equation (6) is included, by making $Ft = t$.

**Note 1.** In certain examples, where $efx$ contains $t$ also, the demonstration plainly teaches us to regard such $t$ as constant. That is, to differentiate $t$ only where it has taken the place of $x$; or rather, to differentiate with respect to $x$, and change $x$ to $t$ afterward.

**Note 2.** To derive one part from another, let $A_n$ denote the $n^{th}$ term of the formula (6) or (8) to commence with the second term; then,

$$A_n = \frac{1}{n} \frac{d}{dx} (efx.A_{n-1})$$

**Example 1.** In the regular application of the formula to equations, Lagrange found that it always gave the least root. Let it then be applied to give the least root of the common quadratic. Here $x = a + bx^2$; by formula (6),

$$x = a + ba^2 + 2b^2a^3 + 3b^3a^4 + 4b^4a^5 + \ldots$$

The convergence of the series evidently depends on the smallness of $ab$.

**Example 2.** It is required to express the third power of the least root of the quintic $x = a + bx^2$.

By formula (8), $Fx = x^3$, $dfx + dx = 3x^2$, $efx = bx^2$.

$$x^3 = a^3 + 3ba^2 + 3b^2a^3 + 13b^3a^4 + 13b^4a^5 + \ldots$$

**Example 3.** To find what number is equal to ten times its natural or hyperbolic logarithm.

In the primary equation, $t = 0$, $x = 0 + 1^x$. By (6),

$$x = 0 + 1 + \frac{0.2}{2!} + \frac{(0.1)^{x-1}}{n!} + \ldots = 1.118325.$$

**Example 4.** If $a$ denote the left hand digit or digits of a number and $b$ those of the logarithm, it is required to find the remaining digits such that $a + x = \log (b + .01x)$, or such that the right hand digits of both number and logarithm shall be alike.

**Lagrange's Theorem for several variables.** Let $efx = h$, $ef'y = h'$, \ldots in the primary equations,

$$x = t + efx, \quad y = t' + ef'y, \ldots$$

The value of $h$, of $h'$, \ldots can each be written out separately by the preceding equation (6). It only remains to substitute these series in the required function (12) after its development in powers of $h$, $h'$, \ldots by Taylor's Theorem for several variables:
(12) \[ F_x, y, \ldots = F(t + h, t' + h', \ldots = F(t', \ldots = F(t, \ldots + \frac{dF}{dt} \cdot h + \frac{dF}{dt'} \cdot h' + \ldots \]
+ \frac{d^2F}{dtdt'} \cdot hh' + \frac{1}{12} \frac{d^2F}{d^2t} \cdot h^2 + \frac{1}{12} \frac{d^2F}{d^2t'} \cdot h'^2 + \ldots.\]

The result being independent of the order of operations, is better stated in (26).

**Laplace's Theorem.** The primary equation may have the more general form,
(13) \[ x = f(t + \epsilon x) = f(t + h).\]
Taking the function \( f \) of each side and developing the powers of \( h \) by Taylor's Theorem,
(14) \[ f\epsilon x = \frac{h}{e} = fff't + \frac{d}{dt} fff't \cdot h + \frac{d^2}{dt^2} fff't \cdot \frac{h^2}{1 \cdot 2} + \ldots.\]

Let this be compared with equations (3) and (5). Again, taking any function of each side of equation (13) preparatory to development by Taylor's Theorem,
(15) \[ F_x = Ff'(t + h).\]

Let this be compared with equation (7). Whence it is obvious that Laplace's Theorem will be found by simply changing \( t \) (8), \( Ft \) into \( fff't \), and \( F \) into \( fff't \). That is,
(16) \[ F_x = fff't + eff't \cdot \frac{dFFF't}{dt} + \frac{1}{1 \cdot 2} \cdot \frac{d}{dt} \left( efFFF' \cdot \frac{d}{dt} fff't \right) + \ldots.\]

In the case of several variables, the changes in equations (12) and (26) are similar and obvious, if the primary equations have each but one unknown quantity as in (11).

The more general equations will be investigated presently.

**Convergence.** To render Lagrange's series more convergent, let \( a \) denote any approximate value of \( x \) or \( \epsilon x \); as \( a = b \epsilon ft \), where \( b = 1 + d(\epsilon ft) / dt \), nearly. Then if \( t' = t + a \), the primary equation becomes
\[ x = t' + (\epsilon ft - a); \]
hence,
(17) \[ F_x = Ft' + (\epsilon ft - a) \cdot \frac{dFtt'}{dt'} + \frac{1}{1 \cdot 2} \cdot \frac{d}{dt'} \left\{ (\epsilon ft - a) \cdot \frac{dFtt'}{dt'} \right\} + \ldots.\]

Again, the arbitrary nature of \( F_x \) or \( Ft \) can generally be made to give a more convergent series. For illustration, let \( Fx \) in equation (8) have the form below, where \( m \) denotes any assumed positive integer; and \( \epsilon ft = Tdt \),
(18) \[ \frac{1}{(Fx)^m} = \frac{1}{(it)^m} - \frac{et}{1 \cdot 2} \cdot \frac{d}{dt} \left( \frac{mT}{it} \right) - \frac{et}{1 \cdot 2 \cdot 3} \cdot \frac{d^2}{dt^2} \left( \frac{mT}{it} \right) \ldots \]
- \[ \frac{et+1}{1 \cdot 2 \cdot (m+1)} \cdot \frac{d^m}{dt^m} \left( \frac{mT}{it} \right) - \frac{et+2}{1 \cdot 2 \cdot 3 \cdot (m+2)} \cdot \frac{d^m+1}{dt^m+1} \left( \frac{mT}{it} \right) \ldots.\]
Observe whenever \( T = ft \), one term vanishes by differentiation.
Again, \( Fx \) may take the forms below, if integrable:

\[
\int \frac{dx}{f(x)} = \int \frac{dt}{f(t)} + e + \frac{e}{1.2} \frac{d(e^t)}{dt} + \frac{e}{1.2.3} \frac{d^2(e^t)}{dt^2} + \ldots
\]

\[
\int \frac{dx}{(f(x))^2} = \int \frac{dt}{(f(t))^2} + e + \frac{e}{1.2.3} \frac{d^2(e^t)}{dt^2} + \frac{e}{1.2.3.4} \frac{d^3(e^t)}{dt^3} + \ldots
\]

Example. Kepler's Problem. Applying the last formula to (20) so well known in Astronomy, we obtain (21).

\[
(20) \quad u = t + e \sin u, \quad \int \frac{du}{\sin u} = -\cot u.
\]

\[
(21) \quad \cot u = \cot t - e + \sin t + \frac{1}{3}e^3 \sin t + \frac{1}{6}e^4 \sin 2t - \ldots
\]

Or multiplying by \( \sin u \sin t \) and reducing,

\[
(22) \quad \sin (u - t) = e' \sin u.
\]

Whence

\[
(23) \quad \tan (u - \frac{1}{3}t) = \frac{1 + e'}{1 - e'} \tan \frac{1}{3}t.
\]

\[
(24) \quad e' = e - \frac{4}{5}e^3 \sin^2 t - \frac{1}{3}e^5 \sin^4 t \cos t + e' (\frac{1}{4} \sin^3 t - \frac{1}{4} \sin^5 t) + \ldots
\]

The limiting values of this series are remarkably simple. For, taking the sine of each side of (20) after transposing \( t \); multiplying and dividing by \( \sin u \); and comparing with (22), we find \( \sin (e \sin u) + \sin u = e' \). Making \( \sin u = 1 \), we have \( \sin e = e' \), the least value of \( e' \). Again, making \( \sin u \) very small or 0, we find \( e = e' \), the greatest value of \( e' \).

Another solution. In (20), and the preceding expression, making \( \sin u = 0, 0.1, 0.2, 0.3, \ldots \) the value of \( e' \) can also be estimated or interpolated from the following

**Outline Table.**

<table>
<thead>
<tr>
<th>( t )</th>
<th>( u - e \sin u )</th>
<th>( 10 )</th>
<th>( \frac{1}{2} )</th>
<th>( \frac{1}{4} )</th>
<th>( \frac{1}{6} )</th>
<th>( \frac{1}{8} )</th>
<th>( \frac{1}{10} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5° 44' - 0.1e</td>
<td>10 sin ( \frac{1}{10} )</td>
<td>( \frac{1}{2} ) sin ( \frac{1}{2} )</td>
<td>( \frac{1}{4} ) sin ( \frac{1}{4} )</td>
<td>( \frac{1}{6} ) sin ( \frac{1}{6} )</td>
<td>( \frac{1}{8} ) sin ( \frac{1}{8} )</td>
<td>( \frac{1}{10} ) sin ( \frac{1}{10} )</td>
<td></td>
</tr>
<tr>
<td>11° 32' - 0.2e</td>
<td>( \frac{1}{2} ) sin ( \frac{1}{2} )</td>
<td>( \frac{1}{4} ) sin ( \frac{1}{4} )</td>
<td>( \frac{1}{6} ) sin ( \frac{1}{6} )</td>
<td>( \frac{1}{8} ) sin ( \frac{1}{8} )</td>
<td>( \frac{1}{10} ) sin ( \frac{1}{10} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17° 27' - 0.3e</td>
<td>( \frac{1}{4} ) sin ( \frac{1}{4} )</td>
<td>( \frac{1}{6} ) sin ( \frac{1}{6} )</td>
<td>( \frac{1}{8} ) sin ( \frac{1}{8} )</td>
<td>( \frac{1}{10} ) sin ( \frac{1}{10} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>23° 35' - 0.4e</td>
<td>( \frac{1}{6} ) sin ( \frac{1}{6} )</td>
<td>( \frac{1}{8} ) sin ( \frac{1}{8} )</td>
<td>( \frac{1}{10} ) sin ( \frac{1}{10} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The value of \( e \) is always positive, also that of \( u \), \( 180° \pm u \), \( 360° - u \).

Symbolic Terms. Expanding the logarithm below in series, developing each numerator by Taylor's Theorem till the power of the increment \( h \) is the same as in the denominator, we see by comparison with (8) that Lagrange's Theorem is represented by the terms independent of \( h \) in the development of the following expression:

\[
(25) \quad Fx = Ft - h \cdot \log \left\{ 1 - \frac{e(t + h)}{h} \right\} \cdot \frac{dF(t + h)}{dt + h}.
\]

This result is equivalent to the definite integral first found in 1805 by M. Perseval in imaginary exponential. Again, if the symbol \( D = d + dt \), \( D' = d + dt' \), \( D^2 = d^2 + dt^2 \) 

formulas (8) and (12) may also be written as in (26).
The index 1 denotes that the first differentiation refers to $F$ only,

$$F_x = e^{D_1 e^t} F_t; \quad F_{x, y, \ldots} = e^{D_1 e^t + D_1 e^t f + \ldots} F_t, f, \ldots$$

**Integration by Series.** Of the two functions contained in Lagrange's full formula, the form of the first, $F_x$, has thus far been fixed by the primary equation (1), while that of $F_x$ may be assumed at pleasure. Conversely for new forms of development, or of integration by series, let the form of $F_x$ be given, while $f_x$ in the primary equation is arbitrary.

Thus let $F_x dx$ denote any expression to be integrated between the limits $x$ and $t$. We assume $f_x$ to be a constant, or an exponential of $x$, or a denominator of $F_x$, or the reciprocal of $F_x$, or to fulfill any advantageous condition. And since the limits $x$ and $t$ are both known, we find $e$ and $e t$ by the primary equation (1), which gives,

$$e = \frac{x - t}{f_x}, \quad e^t = \frac{(x - t)^2}{f_x}.$$  

Substituting this value of $e$ in formula (8), also making $F_x$ and $F_t$ in (8) to represent the required integrals here, and transposing the latter integral,

$$\int F_x dx = \left(\frac{x - t}{f_x}\right) F_t F_t + \frac{1}{1.2}(x - t)^2 \frac{d}{dt} F_t F_t + \frac{1}{1.2.3}(x - t)^3 \times \frac{d^2}{dt^2} F_t F_t + \ldots$$

From this general series, a great many others can be derived by assuming the form of the function $f$. For illustration, let us assume $f_t F_t = 1$, then $f_x F_x = 1$; $(x - t)F_x = e$; whence,

$$\int F_x dx = e + \frac{1}{1.2} e^2 \frac{d}{dt} \left(\frac{1}{F_t}\right) + \frac{1}{1.2.3} e^3 \frac{d^2}{dt^2} \left(\frac{1}{F_t}\right)^2 + \ldots$$

For more development in series without integration, the expression of $e$ in (27) regarded as constant can be substituted in (8), giving what is equivalent to Burmann's Theorem.

For another integral in series, leaving $n$ to be assumed as a whole number or fraction, positive or negative, let

$$f_t = F_t^n, f_x = F_x^n; \quad \text{then} \quad e = (x - t)F_x^{-n};$$

$$\int F_x dx = eF_t^{n+1} + \frac{1}{1.2} e^n \frac{d}{dt} F_t^{n+2} + \frac{1}{1.2.3} e^2 \frac{d^2}{dt^2} F_t^{n+3} + \ldots$$

To develop other series of this kind, will afford an interesting class of exercises for students of the Integral Calculus.

**The General Theorem.**

A review of the preceding pages has unexpectedly led to the more complete formula, toward which the Binomial Theorem and those of Taylor
and Lagrange have progressively tended; the possibility of which was shown by Laplace, with development for one and two variables. The new demonstration appears short, easy and conclusive. Let the primary equations have the form,

\[ x = t + efx, \ y, \ldots \quad y = t' + e'fx, \ y, \ldots \text{ etc.} \]

where \( f, f', \ldots \) denote any given functions of \( x, y, \ldots \) And let

\[ h = efx \ldots ; \quad h_1 = eft' \ldots \text{ Also } h' = e'fx \ldots ; \quad h'_1 = e'ft' \ldots ; \text{ etc.} \]

Then the law of development of any function \( F \) will be

\[ Fx, y, \ldots = Ft + h, t' + h' \ldots = e^{D_1h_1 + D'_1h'_1 + \ldots} \times Ft, t', \ldots \]
\[ = Ft, t', \ldots + D_1h_1F + \ldots + \frac{D^{n-1}_1D_1^1h_1^1}{n!} F + \ldots + \frac{D^{n-1}_1D^{n-1}_1h^{n-1}_1}{n! n'} F + \ldots \]

Here \( D_1 = \frac{d}{dt} ; \quad D_1^2 = \frac{d^2}{dt^2} ; \quad \ldots \quad D'_1 = \frac{d}{dt'} ; \quad D'_1^2 = \frac{d^2}{dt'^2} ; \text{ etc.} \) The index 1 of \( D_1 \) merely denotes that the first differentiation is special. Thus \( D_1^1 = D^{n-1}_1D_1, \text{ and } D_1^1D'_1 = D^{n-1}_1D^{n-1}_1D_1D'_1. \text{ Ft, t', \ldots in the first term is represented elsewhere by } F. \)

While \( D^{n-1} \) is normal, and refers to the whole term before which it is placed, \( D_1 \) is special or initial, and refers only to a part of its term, described as follows: When \( D_1, D'_1, \ldots \) occur separately, the special differentiation refers to \( F \) only. Thus \( D_1h_1F = h_1DF, \text{ and } D'_1h'F = D^{n-1}(h'_1DF). \)

But, when \( D_1, D'_1 \ldots \) are multiplied together, we shall have

\[ D_1D'_1 = DD'F + \frac{dh'_1}{dt} D'F + \frac{dh}{dt} DF. \]

As might be inferred from the left member, this notation is not to introduce new factors, but to show the special or initial differentiation of \( F, h, h' \), in their place in the given term, which thus becomes three terms.

For \( i \) dimensions, with the same signification as to \( F, h_1, \ldots \) the first term is found by the \( i \) differentiations of \( F \), as shown below. For the remaining terms, when \( i \) is 2 or more, we write the combinations or products of the \( i \) quantities \( h_1, h'_1, \ldots h^{(i-1)} \) taken \( i - 1 \) in a set, then \( i - 2 \) in a set, then \( i - 3 \), and so on, ending with 1 in a set. After each combination we write \( D \) as many times as it had letters, with the same accents respectively, and annexing \( F \), to which these differentiations solely refer. Next, we prefix to each combination and referring to it alone, the complementary number of \( D \)s with such accents, that each term shall contain in all \( D, D', D'', \ldots D^{(i-1)}. \) Thus for three dimensions,
(35) \[ D_1 D'_1 D''_1 = DD'D''F + D''h_1 D'D'F + D'h_1 h''_1 DD''F + Dh'_1 h''_1 D'''F + DD'h'_1 D''F + DD''h'_1 D'F + D''h'_1 DF. \]

Note. The result here will be greatly simplified so far as each primary equation is independent of the others, or contains but one variable. Thus if the primary equations all have the form of equations (11) then all the terms containing \( h \) in the right hand members of (34), (35) vanish by differentiation leaving only the first term; so that the result is represented by formula (26).

Elimination of the higher Algebraic Equations. When \( F = z \), all the terms of (34), (35) vanish except the last, where \( DF = 1 \); and the first or special derivation of \( D', D'' \) etc., must refer to \( h_1 \) only. By such reductions we have found from (33) the remarkable result,

(36) \[ x = t + \varepsilon \frac{Dh_1}{D} - 1 \cdot \varepsilon D'_1 h'_1 + D''_1 h''_1 + \ldots. \]

After development, the first differentiation of \( D'_1 \), of \( D''_1 \), \ldots refers to \( h_1 \) only; the first of \( D \) being already performed. Thus in (33) the exponential of \( D \) was expanded separately, and the others together, giving \( x = (1 + D_1 h_1 + \ldots)(1 + D'_1 h'_1 + D''_1 h''_1 + \ldots) t \); whence (36).

Any function of \( x \) can be found in like manner.

Another Demonstration of Lagrange's Theorem. Let the primary equation as in (1) be \( x = t + efz = t + h \). And let Taylor's Theorem be represented by exponential notation, thus;

(37) \[ F_x = F(t + h) = e^{Dh_1} F_t = F_t + DFT_h + \frac{1}{2!} D^2 FT_h^2 h^2 + \ldots \]

(38) \[ h = efz = \varepsilon^2 \cdot eft; \quad h^2 = (eft)^2 \cdot e^{2Dh}; \quad h^n = (eft)^n \cdot e^{2Dh}. \]

The subscripts 1, 2, \ldots will be finally omitted. Their present use is merely to guide the differentiations of Taylor's Theorem; thus \( D \) applies only to \( t \); \( D \) applies only to \( t \); and so on. While \( h = efz \), let \( h = ef_t; \quad h = ef_2; \quad h = ef_3 \); etc. By successive substitutions of \( h \) from equation (38) to (37),

\[ F_x = e_{12} (Dh + D_h + \ldots) F_t + e_{12} (Dh + D_h + \ldots) F_t. \]

(39) \[ F_x = e_{12} (Dh + D_h + D_h + \ldots) \times F_t. \]

Expanding in series, let the general or \( n \)th term be denoted by,

(40) \[ A_n = \frac{1}{n!} \left( \frac{Dh + D_h + \ldots}{1} \right)^n F_t = \frac{1}{n!} \left( \frac{Dh + D_h + \ldots}{1} \right) A_{n-1}. \]
As in geometrical progression each term is found from the preceding by a fixed ratio; so here each term is shown to be derivable from the preceding by a fixed operation. It only remains to ascertain or simplify such operation. For this purpose, omitting the subscripts and applying common substitutions from (38) to (37) we readily find

\[
\begin{align*}
A_0 &= Ft; \\
A_1 &= eft.DFt; \\
A_2 &= \frac{1}{2!} D \left( eft^2.DFt \right) \\
A_3 &= \frac{1}{3!} D^3 \left( eft^3.DF \right) = \frac{1}{3} D \left( eft.A_2 \right) \\
A_4 &= \frac{1}{4!} D^4 \left( eft^4.DF \right) = \frac{1}{4} D \left( eft.A_3 \right) \ldots. \\
A_n &= \frac{1}{n!} D^{n-1} \left( eft^n.DF \right) = \frac{1}{n} D \left( eft.A_{n-1} \right). 
\end{align*}
\]

(41)

Hence the fixed operation in the last member of equation (40) consists in multiplying by \( eft/n \) and taking the derivative of the product. Commencing with \( A_1 \), this is the law of the terms in Lagrange's Theorem.

**Demonstration of the General Theorem.** By Taylor's Theorem, equations (32) become,

\[
h = efx + \ldots = eft + h, t' + h', \ldots = eftt' + \ldots \frac{Dh + D'h'}{2} + \ldots,
\]

\[
h^n = (eft + h, t' + h', \ldots)^n = (eftt' + \ldots)^n \frac{Dh + D'h'}{2} + \ldots.
\]

\[
Fxy \ldots = Ft + h, t' + h', \ldots = Ftt' + \ldots \frac{Dh + D'h'}{2} + \ldots
\]

In \( h, h', \ldots \) of the exponent, substituting \( t + h \) for \( x, t' + h' \) for \( y, \ldots \) and again applying Taylor's Theorem; and so on,

\[
Fxy \ldots = Ftt' + \ldots \frac{\Sigma (D^i h + D^i h')}{i!} + \ldots
\]

(42)

Here the summation extends from \( i = 1 \) to infinity. Developing in series, and denoting the general term by \( A_n \), we find

\[
A_{n+w+\ldots} = \frac{1}{n! n'! \ldots} \left( \Sigma D^i h \right)^n \left( \Sigma D^i h' \right)^{n'} \ldots Ftt' + \ldots
\]

\[
= \frac{1}{n} \left( \Sigma D^i h \right) A_{n-1+w+\ldots}.
\]

To determine this fixed operation, we develop the first terms of formula (42),
and there find it to be always the same as before proved in (40) and (41), except the initial or special differentiation, which was generalized between (34) and (35). This generalization was based on the fixed nature of the operation in connection with the actual developments copied in (34) and (35); which prove the general Theorem, already described.

RECENT RESULTS IN THE STUDY OF LINKAGES.

BY PROF. W. W. JOHNSON, ANNAPOLIS, MD.

Colonel Peaucellier's discovery of an exact rectilinear motion produced by means of jointed rods is interesting not only for its own sake but as having been the starting point of a new and beautiful branch of Geometrical Study. An account of Peaucellier's invention and of some of the earlier results of the Study of Linkages was contributed by the writer to the Analyst for March, 1875; the present article relates to more recent results due principally to the English Mathematicians who have worked in this new field.

The term linkage is employed by Prof. Sylvester to denote a net-work of jointed bars, such that one bar being fixed, any point of another bar will describe a definite locus while the system changes its shape, or undergoes deformation. Thus a jointed quadrilateral is a linkage; if two bars jointed together have their free ends jointed one to each of two bars, the system becomes a six-bar linkage. It is evident that in this way we may form a linkage of any even number of bars, but that we cannot form a linkage of an odd number of bars.

The joints of a linkage constitute a system of points bound by an even number of conditions, each of which asserts the invariability of the distance of two points of the system. A point rigidly connected with one of the bars, whether on or off the straight line connecting the joints, may be regarded as determined by its distance from the two joints, that is by two conditions of the same form: but when a joint is assumed on a bar already included in the linkage, these conditions are not counted in estimating the order of the linkage, which depends not upon the number of joints but solely upon the number of bars which are moveable relatively to one another. The several joints on a single bar are usually taken in a straight line, but Sylvester remarks that "the true view of the theory of Linkages is to consider each bar as carrying with it an indefinitely extended plane."
If one bar of a linkage is fixed in a plane, the joints on the fixed bar become fixed pivots, and the fixed bar may be removed. The system, now consisting of an odd number of bars, is called a linkwork. In this point of view, the study of a linkwork becomes a question of the relative motion of two bars of a linkage.

But the formation of a linkwork for the purpose of describing a given curve is more frequently effected by the employment of a linkage as a transforming cell. When a linkage is thus employed one point called the fulcrum is fixed in a plane, and two other points are regarded, the one as tracer, and the other as follower; then when the tracer describes a given curve the follower describes a curve into which the given curve is said to be transformed by the given cell. Thus the Peaucellier cell (see Analyst, March, 1875) transforms a curve into its inverse. The Pantograph in its simplest form is a four-bar cell, in which the follower describes a curve similar to that described by the tracer.

If the tracer of a cell with a fixed fulcrum be constrained by a linkwork to describe a known curve, we shall have formed a linkwork in which the follower describes the transformed curve. The simplest constraining linkwork is a single bar causing the tracer to describe a circle, hence primary interest attaches to the transformations of the circle, the Peaucellier rectilinear motion, for instance, is a case of "inversion" of the circle.

The distances of the tracer and follower from the fulcrum of a cell are called the rays. In the forms which have been studied the rays are, for the most part, in one straight line. The angle which this line makes with a fixed line in the plane being denoted by $\theta$, and the rays by $r$ and $\rho$, if we have the relation between $r$ and $\rho$ characteristic of the cell, we can readily pass from the polar equation of the curve described by the tracer to that of the curve into which it is transformed.

The simplest relation between $r$ and $\rho$ is $r = m\rho$ as in the Pantograph. No cell in which $r$ and $\rho$ are connected by the general linear relation seems to have been devised; but Messrs Hart and Kempe have given 8-bar linkages in which one point moves always in the line of one of the bars. Such a linkage, the point in question being taken as fulcrum, and the extremities of the bar as tracer and follower, would be a cell in which $\rho - r = a$ constant. The simplest relation of the second degree is $\rho r = a$ constant, in which $r$ and $\rho$ are inversely proportional, as in the Peaucellier cell, and in Hart's reciprocator. (see Analyst, Vol. II, p. 44.)

A six-bar cell, in which $\rho^2 - r^2 = a$ constant, called by Sylvester the Quadratic Binomial Extractor, may be reduced to the four-bar cell represented in Fig. 1, in which $AB = BO = BQ = a$ and $AC = CO = CP = b$. 
The line joining $A$ and $O$ is evidently perpendicular at once to $PO$ and to $OQ$. We therefore have $P$, $O$, and $Q$ always in a straight line, and denoting $PO$ by $\rho$ and $OQ$ by $r$

$$r^2 - \rho^2 = 4(a^2 - b^2). \ldots (1)$$

This cell may be used to extract the square root of a binomial quadratic expression mechanically.

If we take $CP' = mb$, $BO' = BQ' = ma$, $P'O'$ and $O'Q'$ will always be in a straight line; and, denoting them by $\rho$ and $r$, we shall have a more general quadratic cell. Denote the segments into which $CB$ is divided by $AO$ by $v$ and $z$ then

$$z^2 - v^2 = a^2 - b^2. \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (2)$$

But

$$r = 2mx \quad \text{and} \quad \rho = (1 + m)v + (1 - m)z \quad \text{hence}$$

$$z = \frac{r}{2m} \quad \text{and} \quad v = \frac{\rho}{m + 1} - \frac{(1 - m)r}{2m(1 + m)}$$

Substituting in (2) and reducing

$$r^2 + (1 - m)r\rho - m\rho^2 = m(m + 1)(a^2 - b^2) = k^2. \ldots \ldots (3)$$

This is not however a general quadratic relation, but is subject to the condition that the coefficient of the product equals the sum of the coefficients of the squares. It will be observed that it is impossible to convert this into a linear relation for if we impose the condition that the first member shall be a perfect square, we find $m = -1$, but this reduces the equation to $r + \rho = 0$.

As an illustration of transformation, let us apply this cell to the circle

$$\rho = c \cos \theta$$

which passes through the pole or fulcrum. Substituting this value of $\rho$ in (3) we have

$$r^2 + (1 - m)r\cos \theta - m c^2 \cos^2 \theta = k^2,$$

or in rectangular coordinates

$$(x^2 + y^2)^2 + (1 - m)(x^2 + y^2)c x - (m c^2 + k^2)x^2 - k^2 y^2 = 0. \ldots (4)$$

The transformed curve is therefore a bicircular quartic with a node at the origin, the inverse of which with respect to its node is a conic. Hence if (4) be itself transformed by means of a reciprocator attached to the same fulcrum $O'$ the result will be a conic.

Now make $P'$ in Fig. 1 the fulcrum and $P'O' = \rho$, $P'Q' = r$, then $O'Q'$ which is $r$ in equation (3) is denoted by $r - \rho$. Therefore substituting in (3) $r - \rho$ for $r$ we have

$$r^2 - (1 + m)\rho \rho = m(m + 1)(a^2 - b^2) = k^2. \ldots \ldots \ldots \ldots (5)$$
This cell is therefore equivalent to that described on page 43 Analyst Vol. II. If we take \( Q' \) as fulcrum we shall find a similar relation between \( r \) and \( \rho \).

A cell similar to Peaucellier's, except that the symmetrically situated pairs of bars are of the three unequal lengths \( l, m, \) and \( n \), has been called the Scalene Cell. The relation between \( r \) and \( \rho \) is

\[
r\rho (r + \rho) + (m^2 - l^2) r + (m^2 - n^2) \rho = 0. \quad \ldots \ldots \quad (6)
\]

Prof. Cayley has considered the transformation by this cell in an Article in the Quarterly Journal of Mathematics for May 1875; with the results that for a circle passing through the fulcrum, and also for a straight line, the transformed curve is a circular cubic; for the circle in general it is a symmetrical bicircular quartic, but the Cartesian though belonging to the latter class cannot be described in this manner.

A remarkable discovery of Sylvester's is the Plagiograph or Skew Pantograph, of which he gave an account in Nature for July 1st 1875.

Let \( AOBc \) be a jointed parallelogram, and let \( P \) and \( Q \) be so taken that \( AP : PC = BC : CQ \), then \( OPQ \) is a straight line and \( O \) being the fulcrum, \( P \) and \( Q \) are the tracer and follower in the ordinary pantograph. Now take a point \( P' \) rigidly connected with the bar \( AC \), and \( Q' \) rigidly connected with \( BC \), and such that \( AP' = AP \), \( BQ' = BQ \), and the angles \( P'AP \), \( Q'BQ \) are equal but taken in opposite directions. Then \( OAP' \) and \( OBQ' \) are similar triangles and the angle \( P'Q'O \) (difference between \( AOB \) and the sum \( AOP' + AP'O \)) is equal to \( PAP' \) (difference between the supplements \( AOC \) and \( AOP' \)), that is \( P'OQ' \) is constant. Thus the rays \( OP' \) and \( OQ' \) have a constant ratio and make a constant angle, and if \( P' \) describe a given curve, \( Q' \) will describe a similar curve, as in the case of the ordinary pantograph, but turned or skewed through a given angle. The rays of the Skew Pantograph may be made equal by taking \( P \) and \( Q \) to coincide at \( C \).

This extension of the Pantograph suggested at once to Prof. Sylvester and to Mr. Kempe an analogous generalization of Mr. Hart's discovery of a four-bar reciprocator. The heavy lines in Figure 3 represent the four bars of Hart's reciprocator, but the points \( P, P', Q, Q' \), instead of being taken on the bars \( AD, BC, AB \) and \( DC \) respectively (as in Fig. 5, page 44, Analyst Vol. II), are taken off but rigidly connected with these bars at the vertices of similar triangles constructed upon them, as represented by the dotted lines.
in the figure. Now if we join $PQ$, $PQ'$, $P'Q$, $P'Q'$, it is readily seen that
$P'CQ'$ and $BCD$ are similar triangles; in like manner $PAQ$ & $DAB$
are similar, having the same ratio of similitude as the former pair;
hence as $BCD$ & $DAB$ are equal; so likewise are $P'CQ'$ and $PAQ$.
Therefore $P'P = PQ$.

From the triangles $P'DQ'$ and $P'BQ$, which can be shown to be equal in
the same manner, we have $PQ' = P'Q$. Hence $PQP'Q'$ is a parallelo-
gram. By reason of the similar triangles $P'Q'$ bears a fixed ratio to $BD$,
namely that of $Q'C$ to $DC$; and $PQ'$ bears a fixed ratio to $AC$, that of
$DQ'$ to $DC$. Now as proved in the former article the product $BD \times AC$
$= (BC)^2 - (DC)^2$, hence the product $P'Q' \times P'Q' = the difference of the
squares of the unequal links multiplied by the ratio of the product of the sides
of one of the similar triangles to the square of the link on which it is constructed.

$PQP'Q'$ is then a parallelogram the product of whose sides is constant;
it remains to prove that its angles are constant. The angles $DQ'P$ and
$CQ'P'$ being by the similar triangles equal to $DCA$ and $CDB$, their sum
is two right angles; therefore the sum of the remaining angles at $Q'$, namely
$PQ'P'$ and $DQ'C$ is two right angles, that is $PQ'P'$ is the supplement of the
fixed angle $DQ'C$.

Now taking $Q$ as fulcrum, $QP$ and $QP'$ are rays making a constant
angle and inversely proportional to one another, hence if $P$ describe a given
curve $P'$ will describe the inverse curve turned through a given angle
about the fulcrum. This skew reciprocator has been named by Sylvester
the Quadraplane.

If $P$ be constrained by a link $OP$, pivoted at $O$ so that $OQ = OP$, to
move in a circle passing through $Q$, $P'$ will describe a straight line as in
the inversion of the circle by an ordinary reciprocator; but whereas in the
ordinary case this line must be perpendicular to the line of centres $OQ$, by
the use of the Quadraplane the line described may make any desired angle
with the line of centres.

*(To be continued.)*
RATIONAL RIGHT ANGLED TRIANGLES
NEARLY ISOSCELES.

BY ARTEMAS MARTIN, ERIE, PA.

I. Two sides of a rational right angled triangle can not be equal, but they may differ by unity only.

Let \( p, b, h \) be the sides of such a right angled triangle; then we have

\[ h^2 - b^2 = p^2, \]

and, when the hypotenuse exceeds the base by unity,

\[ h = b + 1. \]

By division,

\[ h = \frac{1}{2}(p^2 + 1) \quad \text{and} \quad b = \frac{1}{2}(p^2 - 1). \]

It is obvious that \( p \) may be any odd number; therefore we may take

\[ p = 2n + 1, \]

then

\[ b = 2n^2 + 2n, \quad h = 2n^2 + 2n + 1, \]

which are the sides of the \( n \)th triangle.

Let \( n = 1 \), then 3, 4, 5 are the sides of the least triangle, which we will call the first triangle. Take \( n = 2 \), then the sides of the second triangle are 5, 12, 13. The sides of the third triangle are 7, 24, 25; the sides of the fourth triangle are 9, 40, 41; &c.

The sides of the 80th triangle are 161, 12960, 12961; and the sides of the 624th triangle are 1249, 780000, 780001.

II. 1. When the legs differ by unity, let \( \frac{1}{2}(x - 1) = p \) = perpendicular, \( \frac{1}{2}(x + 1) = b \) = base and \( y = h \) = hypotenuse.

Then

\[ \frac{1}{2}(x - 1)^2 + \frac{1}{2}(x + 1)^2 = y^2; \]

whence

\[ x^2 - 2y^2 = -1; \] \hspace{1cm} (1)

or

\[ (x - y\sqrt{2})(x + y\sqrt{2}) = -1. \] \hspace{1cm} (2)

Let \( e \) and \( f \) be known values of \( x \) and \( y \); then

\[ e^2 - 2f^2 = -1, \] \hspace{1cm} (3)

and

\[ (e - f\sqrt{2})^{2n+1}(e + f\sqrt{2})^{2n+1} = -1. \] \hspace{1cm} (4)

Assume

\[ x - y\sqrt{2} = (e - f\sqrt{2})^{2n+1}, \]

then

\[ x + y\sqrt{2} = (e + f\sqrt{2})^{2n+1}, \]

and we obtain

\[ x = \frac{1}{2}\left\{[(e + f\sqrt{2})^{2n+1} + (e - f\sqrt{2})^{2n+1}]\right\}, \] \hspace{1cm} (5)

\[ y = \frac{1}{2\sqrt{2}}\left\{[(e + f\sqrt{2})^{2n+1} - (e - f\sqrt{2})^{2n+1}]\right\}. \]

where \( n \) may be 0, 1, 2, 3, 4, &c.

It is easily seen that (3) is satisfied by \( e = 1, f = 1 \); therefore

\[ x = \frac{1}{2}\left\{[(1 + \sqrt{2})^{2n+1} + (1 - \sqrt{2})^{2n+1}]\right\}, \] \hspace{1cm} (6)

\[ y = \frac{1}{2\sqrt{2}}\left\{[(1 + \sqrt{2})^{2n+1} - (1 - \sqrt{2})^{2n+1}]\right\}. \]
and the sides of the \( n \)th triangle are

\[
\begin{align*}
\alpha_n &= \frac{1}{2} (x_n - 1) = \frac{1}{2} \left[ (1 + \sqrt{2})^2 + 1 + (1 - \sqrt{2})^2 - 1 \right] \cdots (7) \\
\beta_n &= \frac{1}{2} (x_n + 1) = \frac{1}{2} \left[ (1 + \sqrt{2})^2 + 1 + (1 - \sqrt{2})^2 + 1 \right] \\
\lambda_n &= y_n = \frac{1}{2} \sqrt{2} \left[ (1 + \sqrt{2})^2 + 1 - (1 - \sqrt{2})^2 + 1 \right] \cdots (8)
\end{align*}
\]

Expanding the values in (6) by the Binomial Theorem, we get

\[
\begin{align*}
x_n &= (2n + 1)2^n + \left[ \frac{(2n + 1)(2n - 1)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \right] 2^{n-3} + \cdots + 1 \\
y_n &= 2^n + \left[ \frac{(2n + 1)(2n - 1)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \right] 2^{n-3} + \cdots + (2n + 1)
\end{align*}
\]

The operation of involution is very tedious, except when \( n \) is a small number.

2. We have from (1), \( x_n = \sqrt{2} y_n^2 - 1 \). When \( x_n \) and \( y_n \) are very large numbers, the 1 under the radical may be omitted without sensible error, and we have for the superior limit of their ratio \( x_n + y_n = \sqrt{2} \); and the values of \( x_n \) and \( y_n \) are the numerators and denominators of the odd convergents to the square root of 2 expanded as a continued fraction.

\[
\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}} \quad \text{c.c.}
\]

The successive convergents are

\[
\begin{array}{cccccccc}
1 & 3 & 7 & 17 & 41 & 99 & 239 & 577 & 1386 & 3363 & 8119 & 19601 & 47061 & 112391 & 278681 & 682659 & 1689231
\end{array}
\]

Writing \( \frac{x}{y}, \frac{x_1}{y_1}, \frac{x_2}{y_2}, \frac{x_3}{y_3}, \frac{x_4}{y_4}, \text{c.c.} \) for the odd, and \( \frac{v}{z}, \frac{v_1}{z_1}, \frac{v_2}{z_2}, \frac{v_3}{z_3}, \text{c.c.} \) for the even convergents, we have the relations

\[
\begin{align*}
x_n &= 6x_1 - x \quad \quad x_n = 6x_2 - x_1 \\
y_n &= 6y_1 - y \quad \quad y_n = 6y_2 - y_1 \quad \quad y_n = 6y_3 - y_2 \\
\end{align*}
\]

Also

\[
\begin{align*}
x_n &= 2y_1 + x_n-1 \\
y_n &= 2x_n + y_n-1 \\
y_n &= 2z_n + z_n-1
\end{align*}
\]

which are convenient formulas for computing the sides of large triangles.

The sides of the \( n \)th triangle are

\[
2y_n, x_n-1, y_n^2 - z_n^2, y_n^2 + z_n^2
\]

3. When \( n = 1 \), \( x_1 = 7 \), \( y_1 = 5 \), and we have 3, 4, 5, for the sides of the first triangle.
The sides of the second triangle are 20, 21, 29;
of the third, 119, 120, 169;
of the fourth, 696, 697, 985;
of the fifth, 4058, 4059, 5741;
of the 39th, 433729688537240628356005653359, 4337296885372406283560056533560, 613386407933224037990008001809, = y_{39};
of the 40th, 2527961881478169961048032963696, 2527961881478169961048032963697, 3575077977948634627394046618865, = y_{40}.

\[ z_{39} = \frac{1}{2}(y_{40} - y_{39}) = 1480845785007705294702019308528, \]
\[ y_{80} = y_{40}^2 + z_{39}^2, \]
and the sides of the 80th triangle are
10588278309438211127768625972711138460195892610538807320361440
10588278309438211127768625972711138460195892610538807320361441
149740867787388384990496417211933241811765094618559069827415009

4. If the sides of the \( n \)th triangle be given, the sides of the \((n - 1)\)th and \((n + 1)\)th triangles may be found as follows:

Let \( p_n, b_n, h_n \) be the sides of the \( n \)th triangle; \( p_n - d, b_n - d, md - h_n \) the sides of the \((n - 1)\)th triangle; and \( p_n + D, b_n + D, mD - h_n \) the sides of the \((n + 1)\)th triangle.

Then
\[
\begin{align*}
(p_n - d)^2 + (b_n - d)^2 &= (md - h_n)^2, \\
(p_n + D)^2 + (b_n + D)^2 &= (mD - h_n)^2, \\
\end{align*}
\]

from which, since \( p_n + b_n = h_n \),
\[
\begin{align*}
d &= \frac{2mh_n - 2(b_n + p_n)}{m^2 - 2} \\
D &= \frac{2mh_n + 2(b_n + p_n)}{m^2 - 2} \\
\end{align*}
\]

Take \( m = 2 \), then
\[
\begin{align*}
d &= 2h_n - (b_n + p_n) \\
D &= 2h_n + (b_n + p_n) \\
p_n - 1 &= b_n + 2p_n - 2h_n \\
b_n - 1 &= 2b_n + p_n - 2h_n \\
h_n - 1 &= 3h_n - 2b_n - 2p_n \\
\end{align*}
\]
and
\[
\begin{align*}
p_{n+1} &= 2h_n + b_n + 2p_n \\
b_{n+1} &= 2h_n + 2b_n + p_n \\
h_{n+1} &= 3h_n + 2b_n + 2p_n \\
\end{align*}
\]

We have also the relations
\[
\begin{align*}
p_n &= 6p_{n-1} - p_{n-2} + 2 \\
b_n &= 6b_{n-1} - b_{n-2} - 2 \\
h_n &= 6h_{n-1} - h_{n-2} \\
\end{align*}
\]
Hence, by the rules of recurring series, the sides of the \( n \)th triangle are the coefficients of \( r^r, s^s, t^t \) in the expansion of

\[
\frac{3r - r^2}{(1-r)(1-6r+r^2)}, \quad \frac{4s - 7s^2 + s^3}{(1-s)(1-6s+s^2)}, \quad \text{and} \quad \frac{5t - t^3}{1-6t+t^2}
\]

respectively.

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**SOLUTION OF THE GENERAL BIQUADRATIC EQUATION.**

---

BY NEWTON FITZ, NORFOLK, VA.

Assume the identity

\[
x^4 + ax^3 + bx^2 + cx + d = \left[x^3 + \left(\frac{1}{4}a + \sqrt{q}\right)x + p + r\sqrt{q}\right]\left[x^3 + \left(\frac{1}{4}a - \sqrt{q}\right)x + p - r\sqrt{q}\right].
\]

By multiplication we obtain

\[
x^4 + ax^3 + bx^2 + cx + d = x^4 + ax^3 + (2p + \frac{1}{4}a^2 - q)x^2 + (ap - 2rq)x + p^2 - r^2q,
\]

whence

\[
b = 2p + \frac{1}{4}a^2 - q, \quad c = ap - 2rq, \quad d = p^2 - r^2q.
\]

Eliminating \( r \) and \( q \), we have

\[
p^3 - \frac{1}{8}bp^2 + \frac{1}{2}(ac - 4d)p + \frac{1}{8}(4bd - a^2d - c^2) = 0.
\]

Either root of this cubic may be used in the following expressions for the roots of the biquadratic

\[
x' = -\frac{1}{4}a + \frac{1}{2}\sqrt{q} + \frac{1}{2}\sqrt{\left[\frac{1}{4}a^2 + q - 4p + (a - 4r)\sqrt{q}\right]},
\]

\[
x'' = -\frac{1}{4}a + \frac{1}{2}\sqrt{q} - \frac{1}{2}\sqrt{\left[\frac{1}{4}a^2 + q - 4p + (a - 4r)\sqrt{q}\right]},
\]

\[
x''' = -\frac{1}{4}a - \frac{1}{2}\sqrt{q} + \frac{1}{2}\sqrt{\left[\frac{1}{4}a^2 + q - 4p - (a - 4r)\sqrt{q}\right]},
\]

\[
x'''' = -\frac{1}{4}a - \frac{1}{2}\sqrt{q} - \frac{1}{2}\sqrt{\left[\frac{1}{4}a^2 + q - 4p - (a - 4r)\sqrt{q}\right]}.
\]

Substituting for \( r \) and \( q \), their values in terms of \( p, a, b, c, \) and \( d \), gives

\[
x = -\frac{a}{4} + \frac{1}{2}\sqrt{\left(3p + a^2 - 4b\right)} \pm \frac{1}{2}\sqrt{\left(\frac{a^2}{2} - 2p - b \pm \frac{4ap + a^2 - 4ab + 4c}{\sqrt{8p + a^2 - 4b}}\right)}.
\]

As \( p \) has, in general, three different values, there will be twelve different expressions for the roots of the biquadratic, but only four different values, consequently there must exist relations of identity and of symmetry as follows: \( p_1, p_2, p_3 \) being the roots of the cubic, and \( f_1, f_2, f_3, f_4 \) representing the four forms of the roots of the biquadratic,

\[
x' = f_1p_1 = f_3p_2 = f_4p_3, \quad x'' = f_2p_1 = f_4p_2 = f_3p_3, \quad x''' = f_3p_1 = f_1p_2 = f_2p_3, \quad x'''' = f_4p_1 = f_2p_2 = f_1p_3.
\]

The general expression for the root of an equation of the \( n \)th degree will contain \( n \) variables, and by substituting for each of these \( n \) variables conjugate quadratic surds having the same radical part we obtain the expression for the roots of an equation whose degree is \( 2n \), as for example,
1st degree \( x = a, \)
2nd " \( x = \frac{1}{2}a \pm \sqrt{b}, \)
4th " \( x = \frac{1}{4}a \pm \sqrt{b} \pm \sqrt{(c \pm d\sqrt{b})}, \)
8th " \( x = \frac{1}{8}a \pm \sqrt{b} \pm \sqrt{c \pm d\sqrt{b}} \pm \sqrt{[e \pm f\sqrt{b} \pm \sqrt{(g \pm h\sqrt{b})}]}, \)
3rd " \( x = \frac{1}{8}a + b \) or \( \frac{1}{8}a - \frac{1}{8}b \pm \sqrt{c}, \)
6th " \( x = \frac{1}{32}a + \frac{1}{8}b \pm \sqrt{c} \) or \( \frac{1}{32}a - \frac{1}{8}b \pm \sqrt{d} \pm \sqrt{(e \pm f\sqrt{d})}. \)
5th " \( x = \frac{1}{64}a + b \) or \( \frac{1}{64}a - \frac{1}{64}b \pm \sqrt{c \pm d \pm \sqrt{(e \pm f\sqrt{d})]}, \)
10th " \( x = \frac{1}{128}a + \&c. \)

To solve an equation of any degree, we form an equation of the required degree from the roots as given above and equate the coefficients thus formed to those of the given equation, and we shall have \( n \) equations containing \( n \) unknown quantities. Eliminating between these, the solution of the final equation containing one unknown quantity will give the solution of the original equation. The degree of the final equation may generally be determined by a priori considerations. Thus, if \( n \) is odd, the degree of the final equation will be \( n \) also, and no progress has been made; but when \( n \) is any power of 2 the forms of the \( n \) roots are symmetrical and we may reasonably expect some analogy in their solutions. The solution of the biquadratic is known to depend on that of the cubic, and similarly that of the octic should depend upon that of the quintic. This would give relations of identity between the roots as follows: \( r, r', r'', \&c., \) being roots of the octic and \( q', q'', \&c., \) roots of the corresponding quintic,

\[
\begin{align*}
r' &= f_1 q' = f_2 q'' = f_3 q''' = f_4 q'''' = f_5 q''''' , \\
r'' &= f_2 q' = f_3 q'' = f_4 q''' = f_5 q'''' = f_6 q''''' , \\
r''' &= \&c. 
\end{align*}
\]

There being 8 root forms, and the five values of \( q \) entering into each, there would be 40 different roots, unless the identities above written are true.

---

ON THE SUM OF THE CUBES OF ANY NUMBER OF TERMS OF ANY ARITHMETICAL SERIES.

BY L. P. SHIDY, U. S. COAST SURVEY.

The following propositions are thought to be original, but mathematicians may be already familiar with them.

1. The sum of the cubes of any number of consecutive terms of any arithmetical series is divisible by the sum of that series.
II. In any arithmetical series whose first term is equal to the common
difference, the sum of the cubes of any number of terms is equal to the pro-
duct of the common difference by the square of the sum of the series.

Let \( a, a + d, a + 2d, \ldots, a + (n - 1)d \) be any arithmetical series, 
and let \( S \) = the sum of \( n \) terms: then

\[
S = \frac{1}{4}n(2a + (n - 1)d); \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1)
\]

and denoting the sum of the cubes of the terms by \( S' \), we have

\[
S' = a^3 + (a + d)^3 + (a + 2d)^3 + \ldots + [a + (n - 1)d]^3.
\]

Expanding, and adding similar terms gives

\[
S' = na^3 + a^2d[3^2 + 6 + 9 + \ldots + 3(n - 1)] + ad^2[3 + 12 + 27 + \ldots + 3(n - 1)^2] \\
+ d^3[1 + 8 + 27 + \ldots + (n - 1)^3]. \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (2)
\]

But \( 3 + 6 + 9 + \ldots + 3(n - 1) = \frac{3}{2}(n^2 - n) \), \( 3 + 12 + 27 + \ldots + 3(n - 1)^2 \) 
\( = 3[1 + 2^2 + 3^2 + \ldots + (n - 1)^2] = (n^2 - n) \) \( (2n - 1) \) 
\( \frac{1}{2}(n^2 - n) \) \( (2n - 1) \), and \( 1 + 8 + 27 + \ldots + (n - 1)^3 = 1^3 + 2^3 + 3^3 + \ldots 
+ (n - 1)^3 = \left( \frac{1}{4}n(n - 1) \right)^2 \). Substituting these values in equation (2) gives,

\[
S' = na^3 + 3a^2d\left[\frac{1}{2}(n^2 - n)\right] + ad^2\left[\frac{1}{2}(n^2 - n)\right] (2n - 1) + d^3\left[\frac{1}{4}(n^2 - n)\right] \]
\]

Performing the operations indicated this becomes,

\[
S' = na^3 + \frac{3}{2}a^2d - \frac{3}{2}na^2d + nd^3ad^2 + \frac{3}{2}nad^2 + \frac{1}{2}n^2d^3 - \frac{1}{2}n^2d^2 + \frac{1}{2}n^2d^3.
\]

Factoring, \( S' = \frac{1}{2}n(2a + nd - d)(a^2 + na - d + \frac{1}{2}n^2d^2 - \frac{1}{2}nd^2) \),
or

\[
S' = \frac{1}{4}n[2a + (n - 1)d](a^3 + nad - ad + \frac{1}{2}n^2d^2 - \frac{1}{2}nd^2), \ldots (3)
\]

in which the factor \( \frac{1}{4}n[2a + (n - 1)d] = S \). Therefore the sum of the cubes of the terms is divisible by the sum of the series, as announced in 
Prop. I.

If we square both members of equation (1) we have

\[
S^2 = (\frac{1}{4}n)^2(4a^2 + 4nad - 4ad + n^2d - 2nd + d^2),
\]

which becomes \( S^2 = (\frac{1}{4}n)^2(n^2d^2 + 2nd^2 + d^2) \) when \( a = d \). Making \( a = d \) in 
equation (3) it becomes \( S' = (\frac{1}{4}n)(n^2d^2 + 2nd^2 + d^2)d = S'd \); which 
proves Prop. II.

\[\text{SOLUTION OF NUMERICAL EQUATIONS OF HIGHER DE-} \]
\[\text{GREES WITH SECONDARY (IMAGINARY) ROOTS.} \]

\[\text{BY PROF. A. ZIELINSKI, C. E., AUGUSTA, GEORGIA.} \]

I. Any algebraic equation of the \( 2n \)th degree, and having \( n \) pairs of sec-
ondary conjugate roots, will have primary (real) coefficients, and its general 
form will be,

\[
z^{2n} + A_1z^{2n-1} + \ldots + A_{2n-1}z + A_{2n} = 0.
\]
The roots of this equation will have the form,
\[
\begin{align*}
x_1 &= a_1 + b_1i, & x_2 &= a_1 - b_1i, \\
x_3 &= a_2 + b_2i, & x_4 &= a_2 - b_2i, \\
&\quad \vdots & \quad \vdots \\
x_{2n-1} &= a_n + b_ni, & x_{2n} &= a_n - b_ni;
\end{align*}
\]
where \( i = \sqrt{-1} \).

If in (1) we introduce for \( x \) one of its values in (2), for instance, \( x_1 = a_1 + b_1i \), we get
\[
(a_1 + b_1i)^2 + A_1(a_1 + b_1i)^{2n-1} + \ldots + A_{2n-1}(a_1 + b_1i) + A_{2n} = 0;
\]
and developing,
\[
\varphi([A], a_1, b_1) + \varphi([A], a_1, b_1)i = 0,
\]
where \([A]\) stands for \( A_1, A_2, \ldots, A_{2n}\).

This equation cannot reduce to zero, unless (3) \( \varphi([A], a_1, b_1) = 0 \) and (4) \( \varphi([A], a_1, b_1) = 0 \); that is, unless (3) and (4), both of the \( 2n \)th degree, containing two unknown quantities each, reduce simultaneously to zero, id est, for the corresponding values of \( a_1 \) and \( b_1 \).

Proceeding as usual in the solution of simultaneous equations, we get
\[
\varphi([A], a_1, b_1) = Q\varphi([A], a_1, b_1) + X(a_1 \text{ or } b_1),
\]
where \( Q \) is the quotient obtained by the division of \( \varphi(\ldots) \) by \( \varphi(\ldots) \); and \( X(a_1 \text{ or } b_1) \) the ultimate remainder, containing one variable only.

From (3), (4) and (5) we get
\[
X(a_1 \text{ or } b_1) = 0;
\]
and this equation gives us finally the value or the values of \( a \) or \( b \), which, introduced in (3) or (4), will give the corresponding value or values of \( b \) or \( a \).

II. Any algebraic equation of \((2n + 1)\)th degree, having \( n \) pairs of secondary conjugate roots, and one single secondary root, will have secondary coefficients, and its general form will be,
\[
\begin{align*}
x^{2n+1} + (A_1 + B_1i)x^{2n} + (A_2 + B_2i)x^{2n-1} + \ldots + (A_{2n} + B_{2n}i)x + (A_{2n+1} + B_{2n+1}i) = 0.
\end{align*}
\]
Its roots will have the form,
\[
\begin{align*}
x_1 &= a_1 + b_1i, & x_2 &= a_1 - b_1i, \\
&\quad \vdots & \quad \vdots \\
x_{2n-1} &= a_n + b_ni, & x_{2n} &= a_n - b_ni, \\
x_{2n+1} &= a_{n+1} \pm b_{n+1}i.
\end{align*}
\]

Eqn. (1) has the form \( M + Ni = 0 \); and reduces to zero evidently only when simultaneously
\[
M = x^{2n+1} + A_1x^{2n} + A_2x^{2n-1} + \ldots + A_{2n}x + A_{2n+1} = 0, \quad \& \quad N = B_1x^{2n} + B_2x^{2n-1} + \ldots + B_{2n}x + B_{2n+1} = 0.
\]
Now it is plain that (1) reduces to zero for the same values of \( x \) which satisfy \( N = 0 \); therefore (1) and \( N = 0 \) have \( 2n \) roots in common; hence the
\((2n + 1)\)th root of \((1)\) will be a quantity the difference between which and \(x\), multiplied by \(N\), gives \((1)\) as result. Dividing therefore \((1)\) by \(N\), we get the single secondary root of \((1)\). It is evident that, if \((1)\) agrees with the hypothesis, \(N\) will be the exact divisor of \((1)\).

Again it is evident, from the general theory, that

\[
\pm B_1 = \pm b_{n+1}; \quad \text{and} \quad \pm \left( A_1 - \frac{B_2}{B_1} \right) = \mp b_{n+1}.
\]

If therefore

\[
x - \left[ \pm \left( A_1 - \frac{B_2}{B_1} \right) \pm B_1 \right] = x - \left( \mp a_{n+1} \pm b_{n+1} i \right)
\]

is an exact divisor of \((1)\), then \(\mp (a_{n+1} \pm b_{n+1} i)\) will be the single secondary root of \((1)\), and dividing \((1)\) by \((3)\) we get \(N = 0\). But if there is a remainder, the roots of \((1)\) have not the form \((2)\).

The equation \(N = 0\), of the \(2n\)th degree, gives the remaining \(2n\) roots, as explained in \(I\).

**Examples.**  
\(A\).  \((1)\) \(X = x^7 + (0 - 5i)x^6 - (6 - 15i)x^5 - (32 + 15i)x^4 + (35 + 205i)x^3 + (236 - 790i)x^2 - (558 - 1190i)x + (468 - 780i) = 0\).

We have \(N = -5x^6 + 15x^5 - 15x^4 + 205x^3 - 790x^2 + 1190x - 780 = 0\) or \(N = x^6 - 3x^5 + 3x^4 - 158x^3 - 238x^2 + 156 = 0\).

Here \(B_1 = b_{n+1} = 5\); \(A_1 - B_2 + B_1 = -a_{n+1} = 3\); and \(a_{n+1} \pm b_{n+1} / -1 = -3 \pm 5i / -1\).

Now, \(X = 0\) is divisible by \(x + (3 - 5i \cdot -)\) only, and therefore \(x_7 = -3 + 3i / -1\) is a root of \((1)\) and we have

\[X + (x - x_7) = N = 0.\]

Resolving \(N = 0\), we see first if there are any primary integral roots, and get \(x_1 = 2; x_2 = 3\). We have then,

\[
N \div (x - x_1)(x - x_2) = x^6 + 2x^5 + 7x^3 - 18x + 26 = 0 = C;
\]

and introducing in this equation \(x = a + b\sqrt{-1}\), we get,

\[
C = a^6 + 2a^5 - a^4(6b^2 - 7) - 6a(b^2 + 3) + (b^4 - 7b^2 + 2b)
\]

+ \([4ab + 6a^2 - a(4b^3 - 14b) - 18b - 2b^3]) / -1 = 0\); whence \((A)\)

\[
b^4 - b^2(6a^2 + 6a + 7) + a^4 + 2a^3 - 7a^2 - 13a + 26 = 0,
\]

\((B)\)

\[-b^3(2a + 1) + 2a^3 + 3a^2 + 7a - 9 = 0,
\]

and by combining these two equations, we get finally,

\((2)\)

\[16a^4 + 48a^3 + 104a^2 + 128a^4 - 35a^2 - 91a - 170 = 0.\]

We find as primary integral roots of \((2)\), \(a' = + 1, a'' = 2\); and introducing these values of \(a\) in \((B)\), we get the corresponding values of \(b\);
\[ b' = \pm 1, \quad b'' = \pm 3; \text{ and the four roots of } C' = 0 \text{ are,} \]
\[ x_3 = 1 + \sqrt{-1}, \quad x_4 = 1 - \sqrt{-1}, \quad x_5 = -2 + 3\sqrt{-1}, \quad x_6 = -2 - 3\sqrt{-1}. \]

Therefore the seven roots of (1) are as follows:
\[ x_1 = 2, \quad x_2 = 3, \quad x_3 = 1 + \sqrt{-1}, \quad x_4 = 1 - \sqrt{-1}, \]
\[ x_5 = -2 + 3\sqrt{-1}, \quad x_6 = -2 - 3\sqrt{-1}, \quad x_7 = -3 + 5\sqrt{-1}; \]
and 
\[ (x_1 + x_2 + x_3 + \ldots + x_7) = A_1 + B_1 \sqrt{-1} = + (0 - 5\sqrt{-1}); \]
\[ (x_1 \cdot x_2 \cdot x_3 \ldots x_7) = A_{2+1} + B_{2+1} \sqrt{-1} = 468 - 780\sqrt{-1}. \]

B. \[ X = x^3 - (7 + 5i)x^2 + (19 + 30i)x - (13 + 65i) = 0. \]

Here \[ N = -5x^2 + 30x - 65 = 0, \text{ or } N = x^2 - 6x + 13 = 0; \]
and \[ B_1 = b_{-1} = + 5, \quad A_1 - B_2 + B_1 = - a_{+1} = -7 + 6 = -1, \]
\[ a_{+1} \pm b_{+1}\sqrt{-1} = 1 \pm 5\sqrt{-1}. \]

\[ X = 0 \text{ is divisible by } [x - (1 + 5\sqrt{-1})] \text{ only, therefore } x^3 = 1 + 5\sqrt{-1} \]
is a root of \( x = 0 \), and we have \( X + (x - x_3) = N = 0. \)

Introducing in this equation \( x = a + bi \), we get
\[ N'' = a_1^2 - b_1^2 - 6a_1 + 13 + (2a_1 b_1 - 6b_1)\sqrt{-1} = 0; \quad \text{and} \]
\[ (A) \quad a_1^2 - b_1^2 - 6a_1 + 13 = 0, \quad (B) \quad 2a_1 b_1 - 6b_1 = 0. \]

(B) gives \( a_1 = 3 \), and \( A \) gives \( b_1 = \pm 2; \) therefore the roots of \( N = 0 \)
are \[ x_1 = 3 + 2\sqrt{-1}, \quad x_2 = 3 - 2\sqrt{-1}, \quad \text{and the three roots of } X = 0 \]
are \[ x_1 = 3 + 2\sqrt{-1}, \quad x_2 = 3 - 2\sqrt{-1}, \quad x_3 = 1 + 5\sqrt{-1}; \]
and 
\[ (x_1 + x_2 + x_3) = -(7 + 5\sqrt{-1}) = A_1 + B_1 \sqrt{-1}, \]
\[ (x_1 \cdot x_2 \cdot x_3) = -(13 + 65\sqrt{-1}) = A_{2+1} + B_{2+1} \sqrt{-1}. \]

---

Note by Artemas Martin.—I have discovered that the formula given by me at the top of page 119, No. 7, Vol. I of the Analyst, holds only when \( n = 2. \)

Since the equation \( \sqrt[n]{a} = \frac{a}{r_m} \left[ 1 - \left( \frac{R_m}{a} \right) \right]^{\frac{1}{2}}, \) where \( R_m = a - r_m^2, \)
is identical, it should be written, to reduce \( r_m \) and \( R_m \) to integers,

\[ \sqrt[n]{a} = \frac{(10)^n a}{(10)^n r_m} \left[ 1 - \left( \frac{(10)^2 (R_m)}{a} \right) \right]^{\frac{1}{2}}. \]

The formula for the \( n \)th root is
\[ \sqrt[n]{a} = \frac{a}{r_m} \left[ 1 - \left( \frac{S_m}{a} \right) \right]^{\frac{1}{2}}, \quad \text{where } S_m = a^{n-1} - r_m; \]
but it does not appear to be of any practical use except when \( n = 2. \)
RECENT MATHEMATICAL PUBLICATIONS.

COMMUNICATED BY G. W. HILL.


Many of our text-books attribute 8 satellites to Uranus; it is however now conclusively established that but 4 have been seen. So far as observation shows, the orbits of all the satellites of these two planets are circular, and those of Uranus lie in one plane. As this valuable paper may not be accessible to many of the readers of the Analyst, we append the final elements arrived at. \( T \) being the fraction of a century after 1850, the position of the common plane of the orbits of the satellites of Uranus with reference to the ecliptic is given by the following inclination and longitude of ascending node

\[
t = 97^\circ.85 - 0^\circ.013 \times T,
\]

\[
\theta = 165^\circ.48 + 1^\circ.40 \times T.
\]

And \( a \) denoting the angle under which the radius vector of the satellite is seen at a distance whose logarithm is 1.28310, \( P \) the sidereal period in mean solar days, and \( u \) the argument of the Uranocentric latitude of the satellite at the epoch 1872, Jan. 0. 0, Washington mean time, we have

<table>
<thead>
<tr>
<th>Satellite</th>
<th>( a )</th>
<th>( P )</th>
<th>( u )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ariel</td>
<td>13'78</td>
<td>2^d.520383</td>
<td>15^o.82</td>
</tr>
<tr>
<td>Umbriel</td>
<td>19.20</td>
<td>4.144180</td>
<td>130.56</td>
</tr>
<tr>
<td>Titania</td>
<td>31.48</td>
<td>8.705897</td>
<td>224.00</td>
</tr>
<tr>
<td>Oberon</td>
<td>42.10</td>
<td>13.463269</td>
<td>148.97</td>
</tr>
</tbody>
</table>

The mass of Uranus which corresponds to these elements is \( \frac{1}{127} \).

In like manner the elements of the satellite of Neptune, \( a \) being for the distance whose logarithm is 1.47814, and \( u \) for the epoch 1874, Jan. 0. 0, Washington mean time, are

\[
i = 145^\circ.12,
\]

\[
\theta = 184^\circ.50 + 1^\circ.40 \times T,
\]

\[
a = 10^\circ.275, \quad P = 5^d.876900, \quad u = 101^\circ.07.
\]

The corresponding mass of Neptune is \( \frac{1}{127} \). The paper contains brief tables founded on these elements.


SOLUTION OF A PROBLEM.

BY ISAAC H. TURRELL, CUMMINSVILLE, OHIO.

THREE circles, radii $a$, $b$, $c$, touch each other externally; required the radii, $a_1$, $b_1$, $c_1$ of three circles described in the space enclosed by them, each touching the other two and two of the given circles.

A solution of this problem can be easily obtained by means of the equation proved at page 24, Vol. II. of the Analyst.

$T^2 = 8a_1 a$, $T$ being the direct common tangent of the circles whose radii are $a$, $a_1$ and centers $S$, $S'$; this being a pair that do not touch each other.

Since the circles $S$, $S'$, touch $B$, $C$, externally, their center of similitude $P$, which is the intersection of their direct common tangents, will be in the radical axis $DR$. If $N$, $N'$, be the points where this common tangent, which is not shown in the figure, touches $S$, $S'$, it is well known that $PR^2 = PN \cdot PN'$. Again, $a = \frac{PN}{a-a_1} = \frac{PN - PN'}{a_1} = \frac{T}{PN'}$, $\cdots$ $PN' = \frac{2a_1 \sqrt{2a_1}}{a-a_1}$.

Similarly $PN = \frac{2a_1 \sqrt{2a_1}}{a-a_1}$, $\cdots$ $PR = \frac{2a_1 \sqrt{2a_1}}{a-a_1}$.

Draw $SD$ perpendicular to the radical axis $DR$; then $DR$ being the altitude of the triangle $CBS$, its value is $\sqrt{[abc(a + b + c)] / 2(b + c)}$.

Also $(a + b)^2 - (b - SD)^2 = DR^2 = (a + c)^2 - (c + SD)^2$.

Again $\sqrt{(PN^2 - SD^2)} = PD = DR - PR$; whence

$$\sqrt{\left[\frac{8a_1^2 a_1}{(a-a_1)^2} + a^2 - \frac{a^2(b+c)}{(b+c)^2}\right]} = \frac{2a_1 \sqrt{abc(a+b+c)}}{b+c} - \frac{2a_1 \sqrt{2a_1}}{a-a_1},$$

an equation of the second degree in $a_1$.

Squaring this equation, reducing, and solving with reference to $a_1$,

$$a_1 = \frac{bc+ac+ab+\sqrt{2abc(a+b+c)}}{bc+2ab+2ac+2\sqrt{2abc(a+b+c)}}$$

whence $a_1 = a$ or $\frac{1}{1 + m - 1 + a}$, where $\frac{1}{m} = \frac{2}{a} + \frac{2}{b} + \frac{2}{c} + 2(\frac{2}{ab} + \frac{2}{bc} + \frac{2}{ca})\frac{3}{2}$

Similarly $b_1 = \frac{1}{1 + m - 1 + b}$, $c_1 = \frac{1}{1 + m - 1 + c}$, the radii required.
The first value of \( a_1 \) is worthy of note; for if it were required to determine the position of four circles \( b, c, b_1, c_1 \) of given magnitude, that touch each other consecutively, so that a circle could be drawn touching all, the equation \( a_1 = a \), shows that each of these circles fulfills this condition.

Again, \( a, b, c \), being the radii of the first, we have the following relations connecting the radii of the circles of the various groups.

\[
\frac{1}{a_1} = \frac{1}{m_a} - \frac{1}{a}, \quad \frac{1}{a_2} = \frac{1}{m_a} - \frac{1}{a_1}, \quad \frac{1}{a_x} = \frac{1}{m_x} - \frac{1}{a_{x-1}} - \frac{1}{a_{x-1}}
\]

2nd, \( \frac{1}{b_1} = \frac{1}{m_b} - \frac{1}{b}, \quad \frac{1}{b_2} = \frac{1}{m_b} - \frac{1}{b_1}, \quad \frac{1}{b_x} = \frac{1}{m_x} - \frac{1}{b_{x-1}} - \frac{1}{b_{x-1}} \)

3rd, \( \frac{1}{c_1} = \frac{1}{m_c} - \frac{1}{c}, \quad \frac{1}{c_2} = \frac{1}{m_c} - \frac{1}{c_1}, \quad \frac{1}{c_x} = \frac{1}{m_x} - \frac{1}{c_{x-1}} - \frac{1}{c_{x-1}} \)

Now substituting for \( 1 + a_1, 1 + b_1, 1 + c_1 \) in the third group, their values as given in the second, and carrying the resulting values of \( 1 + a_2, 1 + b_2, 1 + c_2 \) into the fourth, and so on, to the \((x+1)\)th group, we find that

\[
\frac{1}{a_x} + \frac{1}{a} = \frac{1}{b_x} + \frac{1}{b} = \frac{1}{c_x} + \frac{1}{c} = \frac{1}{m_{x-1}} - \frac{1}{m_{x-2}} + \ldots + \frac{1}{m}
\]

when \( x \) is odd, and

\[
\frac{1}{a_x} - \frac{1}{a} = \frac{1}{b_x} - \frac{1}{b} = \frac{1}{c_x} - \frac{1}{c} = \frac{1}{m_{x-1}} - \frac{1}{m_{x-2}} + \ldots - \frac{1}{m}
\]

when \( x \) is an even number.

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A PROBLEM IN SURVEYING.

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BY T. J. LOWRY, M. S., SAN FRANCISCO, CALIFORNIA.

Problem:—Required the positions of the two places of observation \( y \) and \( m \), with reference to three known points \( A, B \) and \( C \), having observed at \( m \) the angles \( AmB \) and \( Bmy \) and at \( y \) the angles \( ByA \) and \( Cym \).

Trig. Analysis:—In the isosceles \( \triangle ABc \) we have the base \( AB \) and \( \angle AeB \) \((=2AmB)\), and hence all the \( \angle s \) to find \( Ae \) or \( Be \).

And in \( \triangle Ade \) are known \( \angle Ade \) \((=180^\circ - AmB)\), side \( Ae \), and \( \angle Ade \) \((=AyB)\) to get \( Ad \) and \( de \). Now in isosceles \( \triangle Age \) having sides \( Ae \) and \( ge \), and \( \angle Age \) \((=\)
2(180° — AmB — Bmy)] we get Ag. Then in Δ gAC are given gA, AC, and \(\angle gAC (= gAe + eAB — BAC)\) to find gC and \(\angle AgC\). And then in isosceles Δ gkC we have gC and \(\angle gkC [= 2(180° — gyC)]\) and hence all the angles, to determine gk (= kC). Now in Δ kge are known kg, ge and \(\angle kge (= kgC + Age — AgC)\) to get ke and <gek.

In Δ ked we know de, ek and <dek (= 360° — gek — ged) and hence all the angles and side dk. And in Δ dky we have all the sides and hence all the angles. In Δ yke are known ke, ky and <eky (= dky + dke) to find ey. Then in Δ Aey having ey, Aε and <Aey (= 360° — yek — gek + Aeg) we find Ay. And in Δ ABy are known Ay, AB and observed <AyB to get yB and <yAB. Now in Δ yCAy we have Ay, AC and <yAC (= yAB — BAC) to find <AyC. Then since <Aym = Oym = AyC, and the <yAm = 180° — Bmy — BmA — Aym, we have in the Δ mAB known the <mAB (= yAB + yAm), the side AB, and the <AmB to find Am and Bm.

Geom. Const.:— Through A and B lay down circles of position containing respectively the angles AmB and AyB. Then from AB at A lay off <BAg = Bmy and at B the <ABg = 180° — AmB — Bmy, and g the point of intersection of the two lines thus drawn will be a secant point of the circle ABr and the right line through m and y. Now through g and C sweep a circle of position containing the observed angle gyC, and y, a second point of this and the position circle ABy, is one of the required places of observation (an approximate knowledge of his position will in general tell the observer whether he was at y or x); then draw the right line yg and m the point where it cuts the circle ABr is the other place of observation.

A more general statement of the Rule given at page 146, vol. I, of the Analyst, for plotting the centre of a circle of position is:— At each end of the line, joining the two observed signals, lay off the difference between the observed angle and 90°, on the same side of this line as the place of observation if the observed angle is less than 90°, but if greater than 90°, on the opposite side. And if the observed angle = 90° then the centre of the position circle is at the middle point of the line joining the signals.

SOLUTION OF PROBLEM 94. (SEE PAGE 199, VOL. II.)

BY PROF. W. W. BEMAN, ANN ARBOR, MICH.

From the figure we easily obtain,

\[
\cos a = \frac{y^2 + (o + x)^2 + (a + s)^2 - r^2}{2(a + s)\sqrt{[y^2 + (o + x)^2]}}, \ldots \ldots \ldots \ldots (1)
\]
\[
\cos \beta = \frac{y^2 + (c+x)^2 + y^2 + (c-x)^2 - 4c^2}{2\sqrt{[y^2 + (c+x)^2]}\sqrt{y^2 + (c-x)^2}} \quad \frac{y^2 + x^2 - c^2}{\sqrt{[y^2 + (c+x)^2]}\sqrt{y^2 + (c-x)^2}}
\]

\[
\cos \gamma = \frac{y^2 + (c-x)^2 + (a-s)^2 - r^2}{2(a-s)\sqrt{y^2 + (c-x)^2}}
\]

Now since \(a + \beta + \gamma = 180^\circ\), we have
\[
2 \cos \alpha \cos \beta \cos \gamma + \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma - 1 = 0 \quad \ldots \ldots (4)
\]

Substituting in (4) the values of \(\cos \alpha\), \(\cos \beta\), and \(\cos \gamma\), as given in (1), (2) and (3), we get:
\[
2(a^2 - s^2)(y^2 + x^2 - c^2)[y^2 + (c + x)^2] + (a + s)^2 - r^2] + (a - s)^2(y^2 + (c - x)^2) + (a + s)^2 + (a - s)^2 + (a + s)^2[y^2 + (c + x)^2] + (a - s)^2 - r^2] + (a + s)^2[y^2 + (c + x)^2]
\]

\[
(y^2 + (c-x)^2 + (a-s)^2 - r^2)^2 + 4(a^2 - s^2)(y^2 + x^2 - c^2) - 4(a - s)^2[y^2 + (c + x)^2] = 0. \quad \text{By a laborious reduction we finally obtain:}
\]

\[
4c^2(x^2 + y^2)^2 = 8acx(x^2 + y^2)^2 + 4(2a^2 - s^2) + c^2(2a^2 + s^2) - a^2(a - s) - ar^n(a + s)(x^2 + y^2) - 16ax^2(x^2 + y^2) + 4[(a^2 - s^2)(r^2 - r^2 - 4as) - 4ascx^2 + 16ascx^2 + \{[2(a^2 - s^2) - r^2(a - s) - r^2(a + s)]^2 - 4c(2a^2 - s^2) + r^2(a - s)^2 + r^2(a + s)^2 + (a^2 + 2s^2)\} + (x^2 + y^2) + 8c[(a^2 - s^2) + r^2(a - s)^2 + r^2(a + s)^2 - 2s^2x^2 + x^2 + 2(a^2 - s^2) - r^2(a - s)^2 + r^2(a + s)^2 - 4as(a^2 - s^2 + c^2)^2x + c^2(a - s) - r^2(a + s) - 2s(a^2 - s^2 - c^2)^2 = 0,
\]

which is the equation of the locus of any point in the line \(DC\).

When \(P\) is the middle point, we have merely to make \(s = 0\) in our equation, giving:
\[
4(x^2 + y^2)^2 + 4(2a^2 - 2c^2 - r^2 - r^2)x + 4(2a^2 + 2c^2 - r^2 - r^2)y^2 + 4x^2 - r^2 + 2c^2 - r^2 - r^2)\]

\[
+ 4c(2a^2 - r^2 - r^2)xy + 4c(r^2 - r^2)x(x^2 + y^2) + (2a^2 - 2c^2 - r^2 - r^2)x^2 + [(2a^2 + 2c^2 - r^2 - r^2) - 16a^2c^2]y^2 + 2c(2a^2 - 2c^2 - r^2 - r^2)
\]

\[
(r^2 - r^2)x + c(r^2 - r^2)^2 = 0.
\]

The terms in the last equation are arranged somewhat differently, in order to show the symmetry of the coefficients.

[The solution of this problem as published in No. 1, p. 28, contains an error in reduction in the equation given in line 3 on said page, which vitiated all the subsequent part of the solution and consequently the conclusion.]

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**Remark.** — Though we dissent from some of the conclusions arrived at by Mr. Adcock in the following solution of 89, yet as Mr. Sivertly himself has come to the conclusion that the published solution is defective, we publish Mr. Adcock's solution, hoping that it will elicit a fuller discussion.—Ed.
BY R. J. ADOCK, MONMOUTH, ILLINOIS.

Let \( m, m' \) be the masses of the spheres \( R \) and \( r \); \( x, y \), the horizontal and vertical coordinates of \( r \)'s centre with reference to \( R \)'s centre when in its initial position, \( \theta = DCG \) = angle between line of centres and vertical at time \( t \), \( \theta_1, \theta_2 \) = angular rotations of \( R \) and \( r \) about centres, \( a = \) initial value of \( \theta \), \( F = \) magnitude and direction of the resultant of the reaction between the spheres at the point of contact \( H \), \( p = \) the magnitude and direction of the normal pressure at \( H \).

From the geometrical relations, \( x = R\theta_1 + (R + r)\sin \theta \ldots (1) \), \( y = (R + r)\cos \theta \ldots (2) \), \( R(\theta - a - \theta_1) = \) extent of arcs bro't into contact, \( R(\theta - a - \theta_1) = r + \theta - a = [(R + r) \theta - a - R\theta_1] + r = \theta_2 \ldots (3) \), \( dx^2 + dy^2 = (R + r)d\theta^2 + R^2d\theta_1^2 + 2R(R + r)\cos \theta d\theta d\theta_1 \ldots (4) \).

There being no loss of vis viva in perfect rolling, \( 2m'g(R + r)(\cos a - \cos \theta) = m' \frac{dx^2 + dy^2}{dt^2} + \frac{3}{2}m'r^2\frac{d\theta_1^2}{dt^2} + \frac{1}{2}mR^2\frac{d\theta^2}{dt^2} = \frac{3}{2}m'(R + r)^2\frac{d\theta_1^2}{dt^2} + \frac{1}{2}(m + m')R^2\frac{d\theta^2}{dt^2} \ldots (5) \); \( F \sin \frac{P}{F} = \) tangential component of \( F \) at \( H \), \( F \cos \frac{P}{F} = p = m'g \cos \theta - m'(R + r)\frac{d\theta_1^2}{dt^2} \ldots (6) \), \( = \) normal pressure at \( H \), \( = \) normal component of weight of \( r \) minus its centrifugal force.

\( F \sin \left( \theta - \frac{P}{F} \right) = \) horizontal component of \( F \) at \( H \), acting to right on lower sphere, also \( = \) horizontal force at \( B \), the lower point of contact. \( F \), acting in direction \( HN \), is the only force communicated by the weight of the upper sphere to the lower, and therefore the lower sphere will tend to move to the right or to the left according as \( HN \) passes to the right or left of \( B \).

By the principle that the angular acceleration of a body about a fixed axis = the moment of the impressed forces divided by the moment of inertia with respect to that axis (Bartlett's An. Mech. 8th ed. p. 248), I have the following four eq'ns: The angular acceleration of \( r \) about its central axis = that which would be produced by the force \( F \) acting upward with the lever arm \( GM \), while the body were retained by a fixed axis through \( G \), hence

\[ F_r \sin \frac{P}{F} + \frac{3}{2}m'r^2 = \frac{d^2\theta_2}{dt^2} \ldots (7) \]; \( FR \left( \sin \frac{P}{F} - \sin (\theta - \frac{P}{F}) \right) + \frac{3}{2}mR^2 = \frac{d^2\theta_1}{dt^2} \ldots (8) \)

For lower sphere about axis through \( B \).
For common rotation of both spheres about axis through centre of lower,
\[ FR \sin \left( \theta - \frac{P}{F} \right) + \left\{ \frac{2mR^3}{2} + m'\left[2s^3 + (R + r)^3\right] \right\} = \frac{d^2\theta_1}{dt^2}, \ldots \ldots \ldots (9) \]
and for rotation of upper sphere about axis through \( H \),
\[ r\left( m'g \sin \theta - m'(R + r)\frac{d^2\theta_1}{dt^2}\right) + \frac{2m'r^3}{2} = \frac{d^2\theta_2}{dt^2}. \ldots \ldots \ldots (10) \]
Eliminating \( F \sin \frac{P}{F} \) and \( F \sin \left( \frac{\theta - P}{F} \right) \) from (8) by (7) and (9),
\[ \frac{2m'r^3}{2} \frac{d^2\theta_2}{dt^2} = \frac{9mR^3 + m'(2r^3 + 5(R + r)^3)}{5R} \frac{d^2\theta_1}{dt^2}. \ldots \ldots \ldots (11) \]
Integrating twice and observing that \( \theta_1 \) and \( \theta_2 \) begin together,
\[ \theta_2 = \frac{(9m + 5m')R^2 + 7m'r^2 + 10m'rR}{2m'rR} \times \theta_1. \ldots \ldots \ldots (12) \]
Hence by (3) \( \theta_1 = \frac{2m'R(R + r)(\theta - a)}{9m + 7m'R^2 + 7m'r^2 + 10m'rR} \ldots \ldots \ldots (13) \]
Substituting in (1) \( \theta_1 \) from (13) and \( \theta \) from (2)
\[ x = 2m'R(R + r)(\theta - a) + [9mR^3 + m'(7R^3 + 7r^3 + 10rR)] + (R + r)\sin \theta + \frac{2m'R^3}{2} \left\{ \cos^{-1} \left( \frac{y}{R + r} \right) - a \right\} + [9mR^3 + m'(7R^3 + 7r^3 + 10rR)] + \sqrt{[(R + r)^2 - y^2]} \ldots (14), \] which is the required equation.

From the figure \( NC : R :: \sin \angle CHN : \sin N \); hence by (7) and (9)
\[ NC : R :: \frac{2m'r^3}{2} \frac{d^2\theta_2}{dt^2} : \frac{2mR^3 + m'[2s^3 + (R + r)^3]}{R} \frac{d^2\theta_1}{dt^2}, \ldots \ldots \ldots \ldots \ldots \ldots \ldots (15) \]
and by (11) \( NC : R :: 9mR^3 + m'[2s^3 + 5(R + r)^3] : 2mR^3 + m'[2s^3 + 5(R + r)^3] \),
in which the first term of second couplet being greater than the second term
by \( 7mR^2 \), \( NC \) is in all cases greater than \( R \), and therefore \( F \) or \( HN \) passes
to the right of the point of contact \( B \), and both spheres in all cases have
their motions on the same side of the origin*.

The value of \( d\theta_1 \) from (13) in (5) gives \( \frac{d\theta_2}{dt^2} \), which in (6), \( P = m'g \cos \theta \)
\[ -(R + r)m'\frac{d\theta_2}{dt^2} = 0, \] gives an equation of the 2nd degree in \( \cos \theta \), from
which one of the values of \( \cos \theta \) gives the point of separation.

By equality of (7) and (10), \( F \sin \frac{P}{F} = \frac{2m'g \sin \theta}{R + r} \), \( \frac{d^2\theta_1}{dt^2} \),
which shows that the tangential component employed in giving rotation to
upper sphere about its central axis = \( \frac{1}{2} \) of the tangential component of the
weight \( m'g \), only when the surface on which the rolling takes place has no
motion.

*We dissent from this conclusion, 1st: Because it may easily be shown, from a different
course of reasoning, that, in certain positions, the spheres will roll in opposite directions;
SOLUTIONS OF PROBLEMS IN NUMBER ONE.

Solutions of problems in number one have been received as follows:
From J. M. Arnold, 99 and 103; Prof. W. W. Beman, 101; Prof. P. E. Chase, 98, 99, 101, 102, 105 and 106; Prof. A. B. Evans, 100, 101, 102, 103, 104 and 105; Dr. H. Eggers, 99, 101 and 103; E. S. Farrow, 98, 99 and 102; G. W. Hill, 102 and 103; H. Heaton, 99, 100, 101, 102, 103 and 105; Prof. W. W. Johnson, 103; Prof. H. T. J. Ludwick, 105; Christine Ledd, 106; Artemas Martin, 102, 104 and 105; Dr. A. B. Nelson, 102 and 105; H. Nevington, 99; O. D. Oathout, 99, 103 and 104; L. Regan, 102 and 103; E. B. Seitz, 99, 100, 101, 102 and 103; Prof. M. C. Stevens, 99 and 102.

98.—"A stream of water moves at the rate of 9 miles an hour, with a fall of one foot per mile; what is the momentum of the water per square foot?"

99.—"Two points are given. Without using a ruler—that is with a pair of dividers only—to determine two other points which, with the first two, form the four vertices of a square."

SOLUTIONS BY PROF. P. E. CHASE, PHILADELPHIA, PA.

\[ M = 1 \times \frac{9 \times 5280}{3600} \times 62 \frac{1}{2}; \quad V = \frac{9 \times 5280}{3600}; \quad MV = 10988. \]

99. Describe a regular hexagon, \( a b c d e f, a, b \) being the given points and \( o \) the centre. With \( a \) and \( b \) as centres describe arcs \( f o, oc \). With \( e \) and \( d \) as centres describe arcs tangent to \( fo, oc \). The tangent points are the points required.

[For want of room we are compelled to defer the remaining solutions till the issue of No. 3, in which we will publish solutions as follows:

Of 101 by Prof. Beman; of 102 by E. B. Seitz; of 103 by H. Heaton; of 104 by Prof. Evans; of 105 by Prof. Ludwick and of 106 by Prof. Chase.]

and, 2ndly: Because in the construction of eqn. (8) the conclusion is virtually assumed; for, let the resultant, \( MN \), cut the radius \( BC \), then because the force \( F \) is necessarily a positive quantity and the angle \( CHN \) is greater than angle \( CH \), we have

\[ \frac{P}{E} \sin (\theta - CHN) - \sin CHN \frac{d^{2}q}{dt^{2}} (8); \frac{q}{m} \cdot \frac{d^{2}q}{dt^{2}} = -\frac{5mR^{2} + m'[2r^{2} + 5(R+r)^{2}]d^{2}q}{5R}. \]

\[ . \quad NC: \quad B :- -5mR^{2} + m'[2r^{2} + 5(R+r)^{2}]: 2mR^{2} + m'[2r^{2} + 5(R+r)^{2}], \]

in which the second term of the second couplet is greater than the first by \( 7mR^{2} \). But because in constructing eqn. (8) it is not known which of the two angles, \( CHN \) or \( CNH \), is the greater, therefore, in assuming that \( CHN \) is the greater, the conclusion arrived at is virtually assumed. Besides, the proportion here obtained is manifestly not true, because it implies a constant angle at \( N \), and we know the initial value of \( N \) is zero when \( a = 0. \) —Ed.
PROBLEMS.

107. By R. J. ADOCK.—If \( c = \sqrt{(a^2 - b^2)} \), show that
\[
\frac{a + c}{a - c} = \left( \frac{c + a - b}{c - a + b} \right)^2.
\]

108. By Prof. A. B. Evans.—Employing the notation of Prob. 100, page 31, show by Finite Differences, or other otherwise, that
\[
\frac{1}{a_x} \pm \frac{1}{b} = \frac{1}{a} \pm \frac{1}{b} = \frac{1}{c_x} \pm \frac{1}{c} = \frac{1}{6} \left( [5 + 2\sqrt{6}]^2 + [5 - 2\sqrt{6}]^2 \pm 2 \left( \frac{1}{a_y} + \frac{1}{b} + \frac{1}{c_x} \right) \right)
+ \frac{1}{2\sqrt{6}} \left( [5 + 2\sqrt{6}]^2 - [5 - 2\sqrt{6}]^2 \right) \left( \sqrt{\frac{2}{ab} + \frac{2}{ac} + \frac{2}{bc}} \right); \text{ where the double sign is to be taken plus when } x \text{ is odd and minus when } x \text{ is even.}
\]

109. By Prof. A. HALL.—Show that the determinant
\[
\begin{vmatrix}
  a & b & c & d \\
  b & a & d & c \\
  c & d & a & b \\
  d & c & b & a \\
\end{vmatrix}
\]
is divisible by \((a + b)^2 - (c + d)^2\); and by \((a - b)^2 - (a - d)^2\).

110. By E. B. Seitz.—A circle, radius \( r \), is placed at random on another equal circle. Prove that the average area of the greatest ellipse that can be inscribed in the area common to the two circles, is \( \pi r^2 \left( \frac{1}{2} \pi - \frac{1}{2} \right) \).

PUBLICATIONS RECEIVED.—We have received various Educational Periodicals for which we tender our thanks, though our space will not permit us even to name them in detail. We desire however to call the attention of our readers, especially of engineers and surveyors, to ENGINEERING NEWS; a publication issued at Chicago, by Geo. H. Frost, Editor and Publisher.

ENGINEERING NEWS commenced its third Vol., Jan. 1st, 1876, as a Weekly, having been published two years as a Monthly. It will be found to be a really interesting and valuable publication, and, dealing with all questions of practical engineering, it should be in the hands of every practical engineer and surveyor.

ERRATA.

On page 35, line 9, from bottom, transpose \( a, b \).

" " 37, " 17, after the word expression, insert, of \( \epsilon' \).

" " 37, " 10, from bottom, change \( e \) to \( \epsilon' \).

" " 38, " 4, change \( F_k \) into \( f_k \).

" " 38, " 6, 7, from bottom, change \( F_k \) and \( F_l \) to \( (F_k) \) and \( (F_l) \).

" " 45, " 13, from bottom, for \( AOC' \) read \( OAC' \).

" " 45, " 12, " " \( AOP' \) " \( OAP' \).

" " 49, " 24, should be, \( p_n^2 + b_n^2 = A_n^2 \).
DEMONSTRATION OF THE DIFFERENTIAL EQUATIONS EMPLOYED BY DELAUNAY IN THE LUNAR THEORY.

BY G. W. HILL.

The method of treating the lunar theory adopted by Delaunay is so elegant that it cannot fail to become in the future the classic method of treating all the problems of celestial mechanics. The canonical system of equations employed by Delaunay is not demonstrated by him in his work, but he refers to a memoir of Binet inserted in the *Journal de l' Ecole Polytechnique*, Cahier XXVIII. Among the innumerable sets of canonical elements it does not appear that a better can be selected. These equations can be established in a very elegant manner by using the properties of Lagrange's and Poisson's quantities \((a, b)\) and \([a, b]\). But a demonstration, founded on more direct and elementary considerations, is, on some accounts, to be preferred.

Let \(a\) denote the mean distance, \(e\) the eccentricity, \(i\) the inclination of the orbit to a fixed plane, \(t\) the mean anomaly, \(g\) the angular distance of the lower apsis from the ascending node, \(\lambda\) the longitude of the ascending node measured from a fixed line in the fixed plane, \(\mu\) the sum of the masses of the bodies whose relative motion is considered, and \(R\) the ordinary perturbative function augmented by the term \(\frac{\mu^2}{2L^3}\). Then if we put \(L = \sqrt{\mu a}\),

\[
G = \sqrt{\mu a(1 - e^2)}, \quad H = \sqrt{\mu a(1 - e^2) \cos i},
\]

Delaunay's equations are

\[
\frac{dL}{dt} = \frac{dR}{dl}, \quad \frac{dG}{dt} = \frac{dR}{dg}, \quad \frac{dH}{dt} = \frac{dR}{dH},
\]

\[
\frac{dl}{dt} = -\frac{dR}{dl}, \quad \frac{dg}{dt} = -\frac{dR}{dg}, \quad \frac{dh}{dt} = -\frac{dR}{dH}.
\]

In terms of rectangular coordinates

\[
R = \frac{\mu^2}{2L^3} + \frac{m'}{\left[(x' - x)^2 + (y' - y)^2 + (z' - z)^2\right]^2} - \frac{m'((xx') + (yy') + zz')}{r^4}.
\]
In this expression for $x$, $y$, $z$ ought to be substituted their values deduced from the formulas of elliptic motion, and expressed in terms of $L$, $G$, $H$, $l$, $g$, $h$. It should be noted that the term $\frac{\mu^2}{2L^3} = \frac{\mu}{2a}$ of the zero order with respect to the disturbing forces, has been added to $R$ only to preserve in the equations the canonical form: it is only by amplifying the signification of the word that $l$ can be called an element, as it is not constant in elliptic motion but augments proportionally to the time and $\frac{dl}{dt} = n = \frac{\mu^2}{2L^3}$. It is chosen as a variable in preference to the element attached to it by addition simply to prevent $t$ from appearing in the derivatives of $R$ outside of the functional signs sine and cosine.

The equations
\[
\frac{d^2x}{dx^2} + \frac{\mu x}{r^3} = \frac{dR}{dx}, \quad \frac{d^2y}{dy^2} + \frac{\mu y}{r^3} = \frac{dR}{dy}, \quad \frac{d^2z}{dz^2} + \frac{\mu z}{r^3} = \frac{dR}{dz},
\]
are well known; here, however, $R$ does not contain the term $\frac{\mu}{2L^3}$. By multiplying them severally by $dx$, $dy$, $dz$ adding and integrating is obtained
\[
\frac{dx^2 + dy^2 + dz^2}{2dr^2} - \frac{\mu}{r} + \frac{\mu}{2a} = \int \left(\frac{dR}{dx} \frac{dx}{dt} + \frac{dR}{dy} \frac{dy}{dt} + \frac{dR}{dz} \frac{dz}{dt}\right).
\]
When the elements are made variable this gives
\[
\frac{d}{dt} \left(\frac{\mu}{2a}\right) = -\left(\frac{dR}{dx} \frac{dx}{dt} + \frac{dR}{dy} \frac{dy}{dt} + \frac{dR}{dz} \frac{dz}{dt}\right).
\]
But we have
\[
\frac{dx}{dt} = n \frac{dx}{dl}, \quad \frac{dy}{dt} = n \frac{dy}{dl}, \quad \frac{dz}{dt} = n \frac{dz}{dl},
\]
and hence
\[
\frac{d}{dl} \left(\frac{\mu}{2a}\right) = -n \left(\frac{dR}{dx} \frac{dx}{dl} + \frac{dR}{dy} \frac{dy}{dl} + \frac{dR}{dz} \frac{dz}{dl}\right)
\]
\[
= -\frac{dR}{dl}.
\]
Dividing both members of this equation by $-n = -\sqrt{\mu a^{-3}}$, the left member is seen to be the differential of $\sqrt{\mu a} = L$. Consequently
\[
\frac{dL}{dt} = \frac{dR}{dl}.
\]

Denoting the true anomaly by $\nu$, the orthogonal projection of the radius vector on the line of nodes is $r \cos (\nu + g)$, on a line perpendicular to it and in the plane of the orbit $r \sin (\nu + g)$. And the latter, projected on the plane of reference, is $r \sin (\nu + g) \cos i$, and on a line perpendicular to this plane $r \sin (\nu + g) \sin i$. If the two projections lying in the plane of reference are
again each projected on the axis of \( x \), their sum will be the value of the coordinate \( x \), and the sum of their projections on the axis of \( y \), the value of the coordinate \( y \). Hence
\[
\begin{align*}
x &= r \cos (v + g) \cos h - r \sin (v + g) \cos i \sin h, \\
y &= r \cos (v + g) \sin h + r \sin (v + g) \cos i \cos h, \\
z &= r \sin (v + g) \sin i,
\end{align*}
\]
or substituting for \( i \) its value in terms of \( G \) and \( H \),
\[
\begin{align*}
x &= r \cos (v + g) \cos h - \frac{H}{G} r \sin (v + g) \sin h, \\
y &= r \cos (v + g) \sin h + \frac{H}{G} r \sin (v + g) \cos h, \\
z &= \sqrt{(G^2 - H^2)} r \sin (v + g).
\end{align*}
\]
As \( r \) and \( v \) are functions of \( L, G \) and \( l \) only, the preceding equations show the manner in which \( H, g \) and \( h \) are involved in \( R \).

\( H \) denotes double the areal velocity projected on the plane \( xy \), or
\[
\frac{x dy - y dx}{dt} = H.
\]
Consequently
\[
\frac{dH}{dt} = x \frac{dR}{dy} - y \frac{dR}{dx}.
\]
But the foregoing values of \( x, y, z \) show that we have
\[
\frac{dx}{dh} = -y, \quad \frac{dy}{dh} = x, \quad \frac{dz}{dh} = 0;
\]
and thus
\[
\frac{dH}{dt} = \frac{dR}{dx} \frac{dx}{dh} + \frac{dR}{dy} \frac{dy}{dh} + \frac{dR}{dz} \frac{dz}{dh} = \frac{dR}{dh}.
\]

\( G \) denotes double the areal velocity, and evidently, if for the moment we suppose \( x \) and \( y \) to be drawn in the plane of the orbit, the axis of \( x \) towards the lower apsis,
\[
\frac{dG}{dt} = x \frac{dR}{dy} - y \frac{dR}{dx} = \frac{dR}{dv},
\]
where, in the last \( R \), for \( x, y \) and \( z \) must be substituted their values given above in terms of \( r, v, G, H, g, h \). Now as the only way, in which \( g \) is involved in these values, is by addition to \( v \), it follows that
\[
\frac{dR}{dv} = \frac{dR}{dg};
\]
and this equation is not affected when, for \( r \) and \( v \) in \( R \), are substituted their values in terms of \( L, G \) and \( l \). Consequently
\[
\frac{dG}{dt} = \frac{dR}{dg}.\]
In the elliptic theory
\[
\frac{xdx - zdy}{dt} = \sqrt{(G^2 - H^2)} \cos h,
\]
\[
\frac{ydx - xdy}{dt} = \sqrt{(G^2 - H^2)} \sin h.
\]

Whence we deduce
\[
\frac{d}{dt} \left[ \sqrt{(G^2 - H^2)} \cos h \right] = x \frac{dR}{ds} - z \frac{dR}{dx},
\]
\[
\frac{d}{dt} \left[ \sqrt{(G^2 - H^2)} \sin h \right] = y \frac{dR}{ds} - z \frac{dR}{dy}.
\]

Eliminating \( \frac{d}{dt} \sqrt{(G^2 - H^2)} \) from these equations, we obtain
\[
\frac{dh}{dt} = \frac{z \sin h}{\sqrt{(G^2 - H^2)}} \frac{dR}{dx} - \frac{z \cos h}{\sqrt{(G^2 - H^2)}} \frac{dR}{dy} - \frac{x \sin h - y \cos h}{\sqrt{(G^2 - H^2)}} \frac{dR}{ds}.
\]

Comparing the coefficients of the three derivatives of \( R \) in the right member of this equation with the values of \( x, y \) and \( z \) in terms of \( r, v, G, H, g, h \), we recognize that they are severally equivalent to the negative of the partial derivatives of these quantities with respect to \( H \). So that
\[
\frac{dh}{dt} = - \left( \frac{dR}{dx} \frac{dx}{dH} + \frac{dR}{dy} \frac{dy}{dH} + \frac{dR}{dz} \frac{dz}{dH} \right) = - \frac{dR}{dH}.
\]

It is a well known principle in the theory of varying elements, that if we differentiate any function, which is a function of the coordinates and \( t \) only, but expressed in terms of \( t \) and the elements, with respect to \( t \) only in as much as it is explicitly involved, we obtain the correct value. Hence if the differentiation is performed on the supposition that the elements are alone variable, the result should be zero. Applying this to the function \( r \), we get
\[
\frac{dr}{dL} \frac{dL}{dt} + \frac{dr}{dG} \frac{dG}{dt} + \frac{dr}{dl} \frac{dl}{dt} - n = 0,
\]
or
\[
\frac{dr}{dL} \frac{dL}{dt} + \frac{dr}{dG} \frac{dG}{dt} + \frac{dr}{dl} \frac{dl}{dt} - n = 0,
\]

or again
\[
\frac{dr}{dL} \left( \frac{dR}{dr} \frac{dr}{dt} + \frac{dR}{ds} \frac{ds}{dt} \right) + \frac{dr}{dG} \frac{dR}{dg} + \frac{dr}{dl} \frac{dl}{dt} - n = 0.
\]

Whence we derive
\[
\frac{dl}{dt} = n - \frac{dr}{dL} \frac{dR}{dr} - \left( \frac{dr}{dl} \right)^{-1} \left[ \frac{dr}{dL} \frac{dL}{dt} + \frac{dr}{dG} \frac{dG}{dt} \right] \frac{dR}{ds}.
\]

From the expression for \( r \) we can eliminate \( l \) and introduce \( v \) in its place by means of the expression for \( v \) in terms of \( L, G \) and \( l \); the result is the well-known equation
\[ r = \frac{a(1 - e \cos u)}{1 + e \cos v} = \frac{G^2}{\mu \left[ 1 + \sqrt{\left( \frac{L^2 - G^2}{L} \right) \cos v} \right]} \]

And we have
\[
\frac{dr}{dl} = \frac{dr}{dv} \frac{dv}{dl}, \quad \frac{dr}{dl} = \left( \frac{dr}{dL} \right) + \frac{dr}{dv} \frac{dv}{dl},
\]

the parentheses denoting the derivative with respect to \( L \) only insomuch as it enters the preceding equation for \( r \). By making these substitutions, the coefficient of \( \frac{dR}{dv} \) in the expression for \( \frac{dl}{dl} \) becomes
\[
- \frac{dv}{dl} - \left( \frac{dr}{dl} \right)^{-1} \left[ \frac{dr}{dL} \frac{dv}{dl} + \frac{dr}{dG} \right].
\]

From the preceding equation for \( r \), we derive
\[
\frac{dr}{dL} = \frac{\mu r^2 \cos v}{L^2},
\]
also the following is a well known equation in the elliptic theory
\[
\frac{dv}{dl} = \frac{G}{nr^2}.
\]

For obtaining the value of \( \frac{dr}{dG} \), \( u \) being the eccentric anomaly, we have the equations
\[ r = a(1 - e \cos u), \quad l = u - e \sin u. \]
Their differentials give
\[
\frac{dr}{de} = -a \cos u + ae \sin u \frac{du}{de},
\]
\[
0 = (1 - e \cos u) du - \sin u de.
\]
Whence
\[
\frac{dr}{de} = -a \cos u - e \frac{1 - e \cos u}{1 - e \cos u} = -a \cos v.
\]
And
\[
e = \sqrt{\left( \frac{L^2 - G^2}{L} \right)}, \quad \frac{de}{dG} = -\frac{G}{L^2},
\]
\[
\frac{dr}{dG} = \frac{dr}{de} \frac{de}{dG} = \frac{G \cos v}{\mu e}.
\]
By substituting the values, it is found that
\[
\left( \frac{dr}{dL} \right) \frac{dv}{dl} + \frac{dr}{dG} = 0.
\]
In consequence
\[
\frac{dl}{dt} = n - \frac{dr}{dL} \frac{dR}{dr} - \frac{dv}{dL} \frac{dR}{dv} = n - \frac{dR}{dL}.
\]
As $R$ is a function of the coordinates and the time only, we can treat it as we have done $r$. Then
\[
\frac{dR}{dL} \frac{dL}{dt} + \frac{dR}{dt}\left(\frac{dl}{dt} - n\right) + \frac{dR}{dG} \frac{dG}{dt} + \frac{dR}{dg} \frac{dg}{dt} + \frac{dR}{dH} \frac{dH}{dt} + \frac{dR}{dh} \frac{dh}{dt} = 0.
\]
On substituting in this the values of the differentials of the elements which have already been determined, it is seen that all the terms but two mutually cancel each other. And on dividing the result by $\frac{dR}{dg}$, we get
\[
\frac{dg}{dt} = -\frac{dR}{dG}.
\]
By adding to $R$ the term $\frac{\mu^2}{2L^2} = \frac{\mu}{2a}$, its partial derivative with respect to $L$ is augmented by the term $-\frac{\mu^2}{L^2} = -n$, but all the other derivatives are unchanged. In consequence of this addition the value of the differential of $l$ becomes
\[
\frac{dl}{dt} = -\frac{dR}{dL}.
\]
An objection may be made against the preceding method of obtaining the differentials of $l$ and $g$, in that the quantities $\frac{dr}{dl}$ and $\frac{dR}{dg}$, which both periodically vanish, have been employed as divisors. But this objection has force only when it is admitted that the differentials of $l$ and $g$ or the corresponding derivatives of $R$ may be discontinuous. For having proved the truth of the equations for all times, except when the divisors, just mentioned, vanish, it follows, that if both members are continuous, the equations must still hold even for the moments of time when $\frac{dr}{dl} = 0$ or $\frac{dR}{dg} = 0$.

Recent Results in the Study of Linkages.

By Prof. W. W. Johnson, Annapolis, Md.

(Continued from page 48.)

The so-called kite-shaped quadrilateral, that which has two pairs of equal adjacent sides, may be employed as a cell in which the rays are always at right angles. Let $ABCD$ be such a quadrilateral the length of its unequal sides being $a$ and $b$. Take any point on $AB$ for the fulcrum $O$ and draw $OQ$ and $OP$ parallel to the diagonals. It is evident that however the
The figure is deformed these lines are parallel to the diagonals which last are always at right angles. Denoting the diagonals $AC$ and $BD$ by $x$ and $y$, and the segments of $AC$ by $s$ and $t$, we have

$$a^2 - b^2 = x^2 - y^2 = x(z - v),$$
and

$$2(a^2 + b^2) = y^2 + 2z^2 + 2x^2 = y^2 + (z + v)^2 + (z - v)^2,$$
whence

$$x^2 + z^2 + \frac{(a^2 - b^2)^2}{x^2} = 2(a^2 + b^2). \ldots (7)$$

Let $AO = ma$ then $\rho = my$, and $r = (1 - m)x$; substituting in (7) we find

$$\rho^2 + \frac{m^2a^2}{(1-m)^2} + \frac{m^2(1-m)^2(a^2-b^2)^2}{r^2} = 2m^2(a^2 + b^2), \ldots \ldots . (8)$$

a relation in which the constants may be any positive quantities since $m$, $a$, and $b$ are arbitrary.

By this cell the circle

$$\rho = c \cos \theta$$

is transformed into

$$\frac{m^2}{(1-m)^2}(x^2 + y^2 + (c^2 - 2m^2a^2 - 2m^2b^2)x^2 - 2(a^2 + b^2)y^2 + m^2(1-m)^2(a^2 - b)^2 = 0, \ldots \ldots (9)$$
a bicircular quartic symmetrical to both axes.

If $O$, $P$, and $Q$ be taken at the vertices of similar triangles similarly placed on $AB$, $AD$, and $CB$, $OP$ and $OQ$ will make constant angles with the diagonals $BD$ and $AC$, and consequently constant angles with one another. We shall therefore still have a cell in which the rays make a constant angle. The rays will still bear fixed ratios to the diagonals and therefore have a relation similar in form to equation (8).

An interesting application of Peaucellier's cell (see fig. 1, p. 41, Analyst, Vol. II) made by Mr. G. H. Darwin in the mechanical transformation of a constant force into one which varies inversely as the square of the distance from a fixed point; by which he remarks it may be "possible to construct a toy to give an ocular proof of elliptic motion." Let the rays be $r$ and $\rho$, then $r\rho = k^2$; and let $F$ and $\varphi$ be forces in equilibrium applied in the directions $\rho$ and $r$, regarded as positive when they tend to increase these quantities. Then differentiating we have

$$d\rho = -\frac{k^2dr}{r^2},$$

but from the principle of virtual velocities,

$$Fd\rho = -\varphi dr,$$

hence

$$\varphi = k^2Fr^{-2}, \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (10)$$

that is, a constant force $F$ applied to the extremity of $\rho$ and directed away from the fulcrum is equivalent to an attractive force $-\varphi$ applied at the extremity of $r$ and varying inversely as the square of the distance from the fulcrum.
M. Mannheim, in a letter to Prof. Sylvester published in the proceedings of the London Mathematical Society, notices that, for the three bars on one side of the axis of a Peaucellier cell, we may substitute three other bars whose squares differ equally from the squares of the bars they replace; the effect of the change is simply to move the joint where the three bars meet in a line perpendicular to the axis. Replacing again the two bars which meet at the extremity of a ray by others whose squares differ equally from those of the bars they replace, the tracer fulcrum and follower are still in a straight line and the cell is equivalent to the Scalene cell. He then derives geometrically from this cell the results already alluded to as established analytically by Prof. Cayley.

Rotation about a fixed point may be regarded as the simplest variety of link motion, or 1-bar motion, the curves described being all circles. In a 3-bar linkwork, the result of fixing one side of a jointed quadrilateral, two of the bars have simple rotation while the third is said to have distinctive 3-bar motion. So in a 5-bar linkwork at least one bar will have 3-bar motion while one or more may have distinctively 5-bar motion. The curves described by points off the bars seem to be of the same character as those described by points on the bars, differing only in the values of the constants introduced.

The general curve in the case of the 3-bar motion is a sextic, but in two cases it breaks up into a circle and a quartic. The sextic is of course symmetrical to the line joining the pivots, and it has three nodes. When the nodes are all conjugate points, the curve may consist of two disconnected ovals, but generally, the curve may be described by continuous motion. When it breaks up into a circle and quartic the quartic retains one node, and the other two nodes of the sextic are represented by points of contact between the circle and quartic. The cases alluded to are those in which the quadrilateral formed by the bars and the line joining the pivots has two pairs of equal sides, either opposite or adjacent. When the opposite sides are equal the circle is described while the figure is in the form of a parallelogram. The quartic described in this case was first shown by Mr. Hart to be the inverse of a conic [see Analyst for March, 1875].

In the case when the adjacent sides are equal, the circle is described when the equal sides of the figure are brought into coincidence, which permits the two equal bars to revolve about a pivot; and the quartic is described while the figure is in the form of a kite-shaped quadrilateral. M. Mannheim has given a geometrical proof that in this latter case the locus described by a point off the bar, but rigidly connected with it, is always the pedal of a conic, or what is the same thing the inverse of a conic.
Mr. A. B. Kempe pointed out this remarkable circumstance in connection with three-bar motion, viz.; that the shape of the curves described depends solely upon the lengths of the moving bars, without reference to their order. Thus in Fig. 5, O and O' being the fixed pivots, for every curve described by a point on b, there exists a similar curve described by a point on a, when a is the middle bar. Prof. Cayley observed that when once stated this was an easy deduc-
tion from the principle of the pantagraph. Complete the parallelogram OABD, then ODBO' is one of the positions of the linkage when the bars are in the order b, a, c. Take P so that O, P and P' are in a straight line; then P and P' may be regarded as the tracer and follower in an ordinary pantagraph; accordingly P' describes a curve similar to that described by P. If AP = mb, the ratio of similitude is 1:m, when m = 0, P is at A and describes a circle and P' is at infinity, which shows that the ultimate shape of the 3-bar curve when indefinitely increased is circular.

Sylvester states that it was this principle and its demonstration by the pantagraph which led him to the discovery of the skew-pantagraph; the corresponding curves which he found to exist also in the case of points off the bars being similar but not similarly placed, and derivable one from another by the linkage to which he has given the above name. The two cases in which the sextic breaks up into a circle and quartic are connected also by Kempe's principle, which has its application also to linkages of higher orders.

Mr. Kempe was the first to produce rectilinear motion by linkwork, otherwise than on the principle of inversion of the circle. This was effected by means of a 7-bar linkwork of which he published an account in the Mathematical Messenger for Dec., 1874. He has since developed a general principle from which he derives a large number of 7-bar rectilinear motions, of which the previously discovered ones including M. Peaucellier's and Mr. Hart's are special cases, "the inversion property of the last two being" to use his words "so to say, accidental." (See Proceedings of the Royal Society, No. 163, 1875.) His method depends upon the formation of a 6-bar linkage in which the projection upon one of the bars of the line joining two points of the linkage is constant. He first points out that the cosines of the opposite angles in a quadrilateral whose sides are given are connected by a linear relation. Let the sides of the quadrilateral ABCD, Fig. 6, be den-
ted by a, b, c and d, then equating two values of the square of the diagonal
we have \[ a^2 + b^2 - 2ab \cos B = c^2 + d^2 - 2cd \cos D. \] 
Now add to the 4-bar linkage \(ABCD\)
the bars \(B'C'\) and \(C'D'\), in such a manner that \(AB'C'D'\) shall be a quadrilateral similar to \(ABCD\); then the angle \(BD'C'\) is the supplement of \(D\). Take \(P\) on \(BC\) and \(P'\) on \(D'C\) such that \(BP : D'P' = ab : cd\), then the projections \(BN\) and \(D'N'\) will be proportional to \(2ab \cos B\) and \(-2cd \cos D\), and therefore in accordance with equation (11) we shall have \(BN + N'D = \) a constant, hence also \(NN'\) is constant. Since \(BP\) is arbitrary any desired value may be given to \(NN'\), and since the ratio of similitude of the two quadrilaterals is arbitrary \(B\) and \(D'\) may be made to coincide.

Among the various methods in which Mr. Kempe employs this fundamental linkage the following will be intelligible without reproducing the author's analytical work and diagrams.

First, fixing the point \(P'\) as a pivot and adding a bar which will cause \(AB\) to remain always parallel to a fixed direction (for instance, a bar \(AO\) equal and parallel to \(BP\)) \(P\) will move in the straight line \(PN\).

Secondly, fixing the points \(B, D'\) and \(A\) (so that the bar \(AB\) may be removed), taking \(BP\) so that \(NN' = \frac{1}{2}BD'\), and adding bars \(PO, OP'\) such that \(PO = PB, P'O = P'D\), \(O\) will move in the straight line \(AB\).

Thirdly, fixing the same points making \(D'\) coincide with \(B\), and taking \(BP\) so that \(NN' = 0\); we may make \(O\) move in the straight line \(AB\), by taking the bars \(PO, P'O\) so that \((P'O)^2 - (PO)^2 = (P'B)^2 - (PB)^2\); or we may make it move in a straight line perpendicular to \(AB\) at \(B\), by taking the bars so that \(PO = P'B\) and \(P'O = PB\).

Fourthly, the same points being fixed and \(D'\) still coinciding with \(B\), complete the parallelogram \(P'BPR\), if the side \(RP\) be produced to \(O\) making \(PO = PR\) the projection of the broken line \(BPO\) is evidently equal to that of the broken line \(P'B\), but this is constant, hence \(O\) moves in a perpendicular to \(AB\).

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**Note.**—That a revolving ellipsoid of three unequal axes can be in equilibrium is said to be proven in a paper by Mr. Ivory in the *Philosophical Transactions* for 1838, page 57. Cannot some of your subscribers inform us in regard to that paper? Walter Silverly.
THE PROBLEM OF THE PASTURAGE. (SEE MATHEMATICAL MONTHLY, VOL. II, PAGE 82.)

BY ALEXANDER EVANS, ELKTON, MARYLAND.

The 137th, and last of the Miscellaneous questions, in the 3rd part of Emerson's North American Arithmetic, which was published in 1835, is as follows:

"If 12 oxen eat up 3½ acres of grass in 4 weeks, and 21 oxen eat up 10 acres in 9 weeks, how many oxen will eat up 24 acres in 18 weeks; the grass being at first equal on every acre, and growing uniformly?"

In June, 1835, a premium of $50 was offered for the most lucid analytical solution of this problem. A Committee was appointed, which reported 48 answers out of 112 to be correct; and by its chairman P. Mackintosh, awarded the prize to James Robinson, whose answer was 37½⅛. In 1860, a review of Mr. Robinson's solution was presented to the National Teacher's Association at Washington, by the Hon. Finley Bigger, then Register of the Treasury, with three solutions; 37½⅛, 37½⅛ and 21⅛.

This review was printed in the 2nd Vol. of the Mathematical Monthly, p. 82; and was supplemented by an Editorial Note at page 85, containing an excellent solution, with the result 37½⅛.

But neither Mr. Emerson, nor the Committee, nor Mr. Robinson, nor Mr. Bigger, nor the National Teachers' Association alludes, nor do the Editors of the Mathematical Monthly allude to the fact that the question, with the exception of a curious misprint, is taken from the Arithmetica Universalis of Newton, which contains besides, a "lucid analytical solution", as the following extract from Ralphson's translation in the Edition of 1728 will show. — The Solutions are Newton's; as is the Example.

"Arithmetical Questions:" page 79, Problem XI.

"If the number of oxen a eat up the meadow b in the time c; and the number of oxen d eat up as good a piece of pasture e in the time f, and the grass grows uniformly; to find how many oxen will eat up the like pasture g in the time h.

"If the oxen a in the time c eat up the pasture b; then by proportion, the oxen b/a in the same time c, or the oxen 60/bf a in the time f, or the oxen 60/bh e in the time h will eat up the pasture e; supposing the grass did not grow at all after the time c.

"But since by reason of the growth of the grass, all the oxen d in the time f can eat up only the meadow e, therefore that growth of the grass in
the meadow e in the time f — c will be so much as alone would be sufficient to feed the oxen d — \frac{eca}{bf} the time f, that is as much as would suffice to feed the oxen \frac{df}{h} — \frac{eca}{bh} in the time h.

"And in the time h — o, by proportion so much would be the growth of the grass as would be sufficient to feed the oxen \frac{h}{f} — o into \frac{df}{h} — \frac{eca}{bh} or
\[
\frac{bdfh - ecah - bdef + eacc}{bfh - bch}
\]

Add this increment to the oxen \frac{eacc}{bch}, and there will come out
\[
\frac{bdfh - ecah - bdef + eca}{bdfh - bch}
\]
the number of oxen which the pasture e will suffice to feed in the time h.

"And so in proportion the meadow g will suffice to feed the oxen
\[
\frac{bdfgh - ecagh - bdegf + eefg}{befh - boeh}
\]
during the same time h.

"Example. If 12 oxen eat up 3\frac{1}{2} acres of pasture in 4 weeks, and 21 oxen eat up 10 acres of like pasture in 9 weeks; to find how many oxen will eat up 24 acres in 18 weeks? Answer 36: for that number will be found by substituting in
\[
\frac{bdfgh - ecagh - bdegf + eefg}{befh - boeh}
\]
the numbers 12, 3\frac{1}{2}, 4, 21, 10, 9, 24, and 18, for the letters a, b, c, d, e, f, g, h respectively; but the solution perhaps will be no less expedite, if it be brought out from the first principles, in form of the precedent literal solution.

"As if 12 oxen in 4 weeks eat up 3\frac{1}{2} acres, then by proportion 36 oxen in 4 weeks, or 16 oxen in 9 weeks, or 8 oxen in 18 weeks, will eat up 10 acres, on supposition that the grass did not grow.

"But since by reason of the growth of the grass 21 oxen in 9 weeks can eat up only 10 acres, that growth of the grass in 10 acres for the last 5 weeks will be as much as would be sufficient to feed the excess of 21 oxen above 16, that is 5 oxen for 9 weeks, or what is the same thing, to feed \frac{1}{4} oxen for 18 weeks.

"And in 14 weeks (the excess of 18 above the first 4) the increase of the grass by analogy, will be such, as to be sufficient to feed 7 oxen for 18 weeks; for it is 5 weeks : 14 weeks :: \frac{1}{4} oxen : 7 oxen.
"Wherefore add these 7 oxen, which the growth of the grass alone would suffice to feed, to the 8, which the grass without growth after 4 weeks would feed, and the sum will be 15 oxen.

"And, lastly, if 10 acres suffice to feed 15 oxen for 18 weeks, then, in proportion, 24 acres would suffice 36 oxen for the same time." So far Newton.

Thus, it will be perceived that Newton had given a "lucid analytical solution" of this problem, about a century and a half before the prize of $50 was awarded to Mr. Robinson: it will also be seen that Newton's example, and that given by Mr. Emerson, have the same identical quantities and figures, except, that the latter has by some means, purposed or accidental, substituted in his copy of Newton's example, 3½ acres for 3½ acres, whereby the somewhat ludicrous result is reached in the solution, that 1 ¼ of an ox is at pasture for 18 weeks. Newton's data, as will be seen, give a whole number of oxen; to wit, 36 oxen.

We do not say that a player must make 24.6 throws with two dice, in order that he may undertake, on an equality of chance, to throw aces once; the a of a throw being as impossible as the grazing of the 1 ¼ of an ox; but we say that the probability of throwing aces once in 24 throws with two dice is less than ¼, and the probability of throwing aces once in 25 throws is more than ¼.

The 1st Edition of the Universal Arithmetic of Newton was published by Whiston in 1707, without Newton's consent, and it is said contrary to his wishes: it appeared at Cambridge, where the original was used in instruction after 1669; or during the time of Newton's Lucasian Professorship. It is proper to state that Whiston declared he had Newton's permission to publish.

The 2nd Edition; with Emendations said to be by Newton, was Machin's in 1722; at London in Latin.

An Edition, called the 2ª: at London; translated by Ralphson, revised and corrected by Mr. Cunn: 1728. This is announced as "very much corrected" and as "carefully compared" with the Edition of 1722 which it calls "the correct Edition of the original." There was also an Edition of 1720, published at London; being, it is supposed, the first Edition of the preceding, or the second of Whiston.

The Leyden Edition of 1732 by S' Gravesande.

notation. Lastly the price and bulk of this book are too great in respect of its utility." Brunei says this Edition is of 1760.

The Rev. Theaker Wilder's Edition; translated by Ralphson, revised and corrected by Cunn; to which is added a treatise upon the measure of ratios by James Maguire. London, 1769. Wilder succeeded Maguire as Professor of Mathematics at Trinity College, Dublin: took the pecuniary risk of the publication, and gave the profits to Maguire's representatives.

Editio, commentariis illustrata et aucta a J. A. Lecci. Milan, 1752, 8vo. 3 vols.


Mr. Wilder mentions, Reyneau, Bernoulli, Maclaurin, Coleson and Campbell as among those persons who had illustrated some particular parts of the work.

The Universal Arithmetic deserves an American Edition, with notes and comments by some mathematician of ability.

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THE SECTION OF A CIRCULAR TORUS BY A PLANE PASSING THROUGH THE CENTER AND TANGENT AT OPPOSITE SIDES.

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BY PROF. E. W. HYDE, CIN. UNIV., CINCINNATI, OHIO.

In the diagram suppose the axes of \( z \) and \( y \) to be in the plane of the paper, and the axis of \( z \) perpendicular to the same.

For any point of the surface we shall have \( x^2 + y^2 = \rho^2 \), and \( \rho = R - \sqrt{r^2 - z^2} \), in which \( R = AC \) and \( r = AB \).

\[
\begin{align*}
\therefore x^2 + y^2 &= R^2 - 2R\sqrt{r^2 - z^2} \\
&+ r^2 - z^2,
\end{align*}
\]

whence by transposing, squaring, reducing and placing \( R^2 - r^2 = \rho^2 \) and \( R^2 + r^2 = q^2 \) we obtain \( (x^2 + y^2 + z^2)^2 - 2q^2(x^2 + y^2) + 2p^2z^2 = -\rho^2 \).

The equation of a plane through the axis of \( y \) and tangent on the opposite sides above and below, is

\[
z = \frac{rx}{\sqrt{(R^2 - r^2)}} = \frac{r}{\rho} z.
\]
Eliminating between the equation of the torus and that of the plane, we obtain
\[ \left( x^2 + y^2 + \frac{r^2}{p^2} x \right)^2 - 2q(x^2 + y^2) + 2r^3x = -p^4, \]
which by expansion and reduction becomes
\[ (R^2x^2 + p^2y^2) - 2R^2px^2 - 2pr^2x = -p^4. \]
This is the equation of an oblique projection of the intersection. To obtain the curve in its own plane, we must substitute for \( x, x' \cos(\text{angle between tangent plane and plane } x \ y) = (p + R)x'. \) Therefore
\[ p'(x'^2 + y'^2) - 2p'x'^2 - 2p'y'^2 = -p^4, \]
or, dividing by \( p^4 \) and dropping primes,
\[ (x^2 + y^2) - 2px^2 - 2qy^2 = -p^4. \]
Add to both sides of this equation \( 2p^2(x^2 + y^2) \), then
\[ (x^2 + y^2)^2 - 2p^2x^2 - 2qy^2 + 2p^2x + 2p^2y^2 = 2p^3(x^2 + y^2) - p^4 \]
or
\[ (x^2 + y^2)^2 - 2y^2(R^2 + r^2 - R^2 + r^2) = 2p^3(x^2 + y^2) - p^4 \]
\[ (x^2 + y^2)^2 - 2p^3(x^2 + y^2) + p^4 = 4r^2y^2. \]
\[ \cdot \cdot \cdot \]
\[ x^2 + y^2 - p^2 = \pm 2ry; \]
\[ \cdot \cdot \cdot \]
\[ x^2 + y^2 \pm 2ry + r^2 = R^2; \]
and
\[ x^2 + (y \pm r)^2 = R^2. \]
Hence the section is two circles whose radius is \( R \), and whose centers are at a distance from the center of the torus equal to the radius of the generating circle of the torus.

In a similar manner we find that when the generating curve is an ellipse, the section is two ellipse whose semi axes are \( R \) and \( \sqrt{(R^2 - a^2 + b^2)} \), \( a \) and \( b \) being the semi axes of the generating ellipse. So with a hyperbolic generatrix, we obtain hyperbolas whose semi axes are \( R \) and \( \sqrt{(R^2 - a^2 - b^2)} \). In this case we must have \( a > R \) in order to render the problem possible. With parabolic generatrix we have a parabola whose parameter (latus rectum) is \( 4(R + p),* \) that of the generatrix being \( 4p \). With the conjugate hyperbola for generatrix the section is imaginary, the equation of its projection being
\[ R^2x^2 + (R^2 + a^2)(y = a \sqrt{-1})^2 = R^2(R^2 - a^2). \]

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**DIFFERENTIATION.**

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BY J. B. MOTT, NEOSHO, MISSOURI.

HAVING seen several explanations of the method of finding the first differential coefficient, I offer the following:

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*In this case \( R \) is the distance from the center of the torus to the vertex of the generating parabola.
In the equation \( u = ax^2 \) (1), let \( x = x + n \) and suppose from the start that the value of \( n \) is infinitely small. There will then be no finite change in the new equation of the values of either \( u \) or \( x \). We shall have

\[
u = a(x + n)^2 = ax^2 + 2anx + an^2.
\]

(2)

Subtracting (1) from (2) gives \( u - u = 2anx + an^2 \) or \( \frac{u - u}{n} = 2ax + an \).

Now as \( n = 0 \), so far as any thing finite is concerned, therefore

\[
\frac{u - u}{n} = 0 = 2ax.
\]

(3)

In the Calculus to preserve some traces of \( u \) and \( x \) from the fraction \( (u - u) + n = 0 + 0 \), authors generally put \( du \) for \( u - u \) and \( dx \) for \( n \), \( d \) being used merely as an abreviation for the word differential, the meaning of which I understand to be infinitely small. According to this notation, from (3) we have

\[
\frac{u - u}{n} = 0 = \frac{du}{dx} = 2ax.
\]

In the same way the following equations; \( u = ax^2, u = ax^2 + bx^3, u = ax + bx^2 + cx^3, u = ax^2 - 4ax^2 + 4x^5, \&c. \), may be differentiated or transformed, respectively, to \( \frac{0}{0} = \frac{du}{dx} = \frac{3ax^2}{3ax^2}, \frac{2ax}{2ax}, \frac{6ax}{6ax}, \frac{4x^2}{4x^2}, \&c. \).

The right hand members in the new equations are obtained most readily by multiplying each term of the primitive by the exponent of \( x \) and diminishing the exponent by unity.

The fraction \( \frac{du}{dx} = \frac{0}{0} \) may come from any quantity, thus \( N = \frac{N}{1} = \frac{N}{1} \times \frac{0}{0} = 0 \), where \( N \) may be any whole number or fraction or 0 or \( \infty \); but there are particular values to this fraction according to the conditions of the problem or equation from which it is derived.

*If by the supposition that \( n \) is infinitely small it is meant that \( n = 0 \), then

\[
u - u' = 2anx + an^2 = 0,
\]

by reason of the zero factor in \( 2anx \) and \( an^2 \), and no logical inference can be drawn from the equation. But if \( n \) is supposed to be some finite quantity, then

\[
\frac{u - u'}{n} = 2ax + an
\]

is an intelligible equation; and it is easily seen that the ratio of \( u - u' \) to \( n \) approaches to \( 2ax \) as \( n \) approaches to zero; from which we infer that the limit of the ratio of \( u - u' \) to \( n \) is \( 2ax \). Therefore “When variable quantities become infinitely great, or infinitely small, their limiting ratio may frequently be determined, though the quantities themselves in such state elude our comprehension.”—Ed.
From the foregoing I infer that, by substituting as above, any equation containing two variables, of the form \( u = Ax^a + Bx^b + Cx^c + \&c. \), may be transformed to \( \frac{du}{dx} = aAx^{a-1} + bBx^{b-1} + cCx^{c-1} + \&c. \)

It may be said that other explanations, by different authors, perform the same transformations. They, however, first consider \( n \) to have some finite value which causes both \( u \) and \( x \) to change their values, but finally they assume \( n = 0 \), so that both \( u \) and \( x \) change back again to what they were at first; making two unnecessary changes as it seems to me.

My explanation supposes the values of \( u \) and \( x \) to remain the same during transformation and not to undergo any change at all; so that differentiation is simply Algebraic transformation; thus showing the connection between Algebra and the Calculus.

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**SOLUTION OF AN INDETERMINATE PROBLEM.**

BY DAVID S. HART, A. M., M. D., STONINGTON, CONN.

[At the conclusion of Dr. Matteson's solution, in Vol. II, page 49 of the Analyst, we alluded to a request that had been made by Prof. Henkle, soon after the publication of Prof. Perkins' solution, that we publish the solution of a similar question that had been made by Dr. Hart some 20 years before, and promised to do so at some future time. As we have recently been reminded by Dr. Hart that his solution had not yet appeared in the Analyst, we subjoin the solution alluded to, with the following prefatory remarks by Dr. Hart.—Ed.]

"This problem was I believe first given by Dr. J. R. Young in his Algebra, published a little more than 60 years ago. The answers he gives are small, but erroneous. Samuel Ward, in his American Edition of Young's Algebra, was the first to detect the error. It is probable that he went no farther than to satisfy himself that the answers must be very large numbers.

"It is now 26 years since I completed a general solution of this problem without conditions annexed. The condition that the squares shall be in arithmetical progression was solved by Prof. Kirkwood in Stoddard and Henkle's University Algebra, but his solution is a particular one involving very large numbers. Prof. Perkins has a general solution in the Analyst, but it is limited to the case above mentioned. He finds the same numbers which I had already found, unknown to him, 25 years before."
“I sent my solution to Dr. Matteson 8 years ago, which he improved and extended so as to be still more general; and this solution, thus improved, was inserted in the Analyst. Dr. M. ought to have given me credit for the general idea of solving all problems of this kind, in doing which I believe I was first in the field.”

**Problem.**—Find three square numbers such, that if each be either increased or diminished by its root, the sums and differences shall be squares.

Let \( a^2x^2, b^2x^2, c^2x^2 \), be the numbers. Then

\[
\begin{align*}
a^2x^2 + ax &= 0, & b^2x^2 + bx &= 0, & c^2x^2 + cx &= 0.
\end{align*}
\]

Let \( a^2x^2 + ax = m^2x^2 \), also let \( a^2x^2 - ax = m^2x^2 \); then

\[
x = \frac{a}{m^2 - a^2} \ldots \ldots (1); \quad \text{or} \quad x = \frac{a}{a^2 - m^2} \ldots \ldots (2)
\]

Using the positive sign

\[
b^2x^2 + bx = \frac{a^2b^2 - a^2b + abm^2}{(m^2 - a^2)^2} = 0, \quad c^2x^2 + cx = \frac{a^2c^2 - a^2c + acm^2}{(m^2 - a^2)^2} = 0;
\]

or, rejecting the square denominators,

\[
a^2b^2 - a^2b + abm^2 = 0, \quad \text{and} \quad a^2c^2 - a^2c + acm^2 = 0,
\]

both of which are squares if \( m = a \). Let \( m = a + n \); then, by substitution, these expressions become

\[
abn^2 + 2a^2bn + a^2b^2 = 0, \quad \text{and} \quad acn^2 + 2a^2cn + a^2c^2 = 0.
\]

Multiplying the first by \( c^2 \) and the second by \( b^2 \), we have

\[
abcn^2 + 2abcn^2 + a^2b^2c^2 = 0 = A^2; \quad abn^2 + 2abcn + a^2b^2c^2 = 0 = B^2.
\]

By subtraction we have

\[
abc(c - b)n^2 + 2abbc(c - b)n = a(c - b)n \times (2abc + bon) = A^2 - B^2 = (A + B)(A - B).
\]

Now putting \( a(c - b)n = A - B \), and \( 2abc + bon = A + B \), and taking half the sum, we have \( A = abc + \frac{1}{2}(ac + bo - ab)n \).

\[
\therefore \quad abc^2n^2 + 2abbc^2n + a^2b^2c^2 = A^2 = [abc + \frac{1}{2}(ac + bo - ab)n]^2.
\]

Reducing this we obtain

\[
n = \frac{4abc(ab + ac - bo)}{(ac + bo - ab)^2 - 4abo}; \quad m = \frac{a[(ac + bo - ab)^2 - 4b(o - ab)]}{(ac + bo - ab)^2 - 4abc^2}.
\]

Substituting this value of \( m \) in (1) and (2), above we finally obtain

\[
x = \frac{\pm [(ac + bo - ab)^2 - 4abc^2]}{8abc(ac + bo - ab)(ac + ab - bo)(ac - ab - bo)}.
\]

Multiplying by \( a, b, c \), we have general expressions for \( ax, bx, \) and \( cx \), the roots of the squares required.

As \( a, b, c \) may be any numbers, they may be taken such that their squares shall be in arithmetical progression, geometrical progression, harmonical
progression; or that the sum of their squares shall be a square, or that they shall be the sides of a right-angled triangle. In fact the number of requirements is unlimited.

In the first case, let \( a = 2rs + r^2 + s^2, b = r^2 + s^2, c = 2rs + r^2 - s^2 \), which are general expressions for three square numbers in arithmetical progression; \( r, s \), being any different numbers that will make \( a, b, c \), all positive. Let \( r = 2, s = 1; \) then \( a = 1, b = 5, c = 7 \), and

\[
x = \frac{151321}{7863240} \cdots \frac{151321}{7863240}' \frac{756605}{7863240} \text{ and } \frac{1059247}{7863240}'
\]

are the roots for the positive sign in \( ax^2 \pm ax = \square \), &c. In order to have a positive value of \( x \) to satisfy the negative sign, take \( r = 4, s = 3 \); then \( a=17, b=25, c=31, \) and

\[
x = \frac{(864571)^3}{11011044931800}.
\]

Therefore

\[
\frac{12707211238697}{11011044931800} \frac{18687075351025}{11011044931800}' \frac{23171973435271}{11011044931800}
\]

are the roots for the negative sign in \( ax^2 \pm ax = \square \), &c.

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**SOLUTIONS OF PROBLEMS IN NUMBER 2.**

101. — "From any point \( O \), within the circumference of a circle, two lines are drawn making a constant angle with each other. These lines revolve about \( O \) in the plane of the circle, and from the points where they cut the circumference tangents are drawn. Find the locus of the intersection of these tangents.

**SOLUTION BY PROF. W. W. BEMAN, ANN ARBOR, MICHIGAN.**

Let a diameter through \( O \) represent the axis of \( X \). Put \( r = \) the radius of the given circle, \( a = \) the distance of the point \( O \) from its centre, \( a = \) the constant angle included by the two given lines; \( x, y \), coordinates of the intersection of the tangents, the origin being at the centre of the circle, and \( x', y' \), and \( x'', y'' \), coordinates of the points where the given lines cut the circumference of the given circle. Then we easily obtain:

1. \( xx' + yy' = r^2 \) \( \cdots (1) \); \( xx'' + yy'' = r^2 \) \( \cdots (2) \);
2. \( x'^2 + y'^2 = r^2 \) \( \cdots (3) \);
3. \( x''^2 + y''^2 = r^2 \) \( \cdots (4) \);
4. \( (x' - x'')^2 + (y' - y'')^2 = (x' - x)^2 + y'^2 + (x'' - x)^2 + y''^2 \)
5. \( -2 \sqrt{\left[\left(\frac{(x' - x)^2 + y'^2\right]}{\left(\frac{(x'' - x)^2 + y''^2\right]}\cos a; \text{ or,}
6. \( x'x'' + y'y'' - a(x' + x'') + a^2 = \sqrt{(r^2 + a^2 - 2ax)(r^2 + a^2 - 2ax')}\cos a \cdots (5)\)

Combining (1) and (3),

\[
x' = \frac{r}{x^2 + y^2} \left( rx \pm y \sqrt{(x^2 + y^2 - r^2)} \right); \ y' = \frac{r}{x^2 + y^2} \left( ry \mp x \sqrt{(x^2 + y^2 - r^2)} \right).
\]
Combining (2) and (4),

\[ x'' = \frac{r}{x^2 + y^2} \left( x^2 + y^2 r^2 \right) ; \quad y'' = \frac{r}{x^2 + y^2} \left( y^2 \pm xy \right) \left( x^2 + y^2 - r^2 \right) \]

Substituting these values in (5), and reducing,

\[ [2r^4 - (r^2 - a^2)(x^2 + y^2) - 2ar^2x]^2 = [(r^2 + a^2)(x^2 + y^2)^2 - 4ar^2(r^2 + a^2)(x^2 + y^2)x + 4a^2r^2(x^2 - y^2)(x^2 + y^2)] \cos^2 a \]

or, in a better form for discussion,

\[ [2r^4 - (r^2 - a^2)(x^2 + y^2) - 2ar^2x]^2 \tan^2 \alpha = 4r^2(x^2 + y^2 - r^2)(ax - r^2) \]

When \( \alpha = 0 \), we have \( x^2 + y^2 - r^2 = 0 \), and \( (ax - r^2)^2 = 0 \), the original circle, and two coincident straight lines. When \( \alpha = 90^\circ \), \[ 2r^4 - (r^2 - a^2) \times (x^2 + y^2) - 2ar^2x^2 = 0 \], two coincident circles.

As \( \alpha \) increases from 0, the circle passes into an oval with the horizontal axis the longer, while the two straight lines separate at first into two infinite branches. The right hand branch recedes to infinity and then reappears on the left, forming now with the other line an oval whose vertical axis is the longer. These ovals then approach till, when \( \alpha = 90^\circ \), they coincide in a circle.

When \( \alpha = 0 \), the equation becomes

\[ (x^2 + y^2 - r^2 \cos^2 \frac{\alpha}{2})(x^2 + y^2 - r^2 \sec^2 \frac{\alpha}{2}) = 0 \]

two circles, coincident when \( \alpha = 90^\circ \), as should be expected.

When \( \alpha = r \), the equation becomes \( (x^2 + y^2 - r^2 \sec^2 \alpha)(x - r)^2 = 0 \), a circle and two coincident straight lines.

The locus can be easily constructed for a number of values of \( \alpha \) at the same time, and the interesting changes seen at a glance.

[Dr. Eggers gets]

\[ 2(px + qy - r^2) \left( x^2 + y^2 - r^2 \right) = 2(px + qy - r^2) \left( r^2 - p^2 - q^2 \right) (x^2 + y^2) \tan \alpha \]

for the required equation; in which \( p \) and \( q \) are coordinates of the point \( O \).

[Mr. Seitz gets]

\[ \cos \theta = \frac{r^2}{\rho^2} - \frac{r^2 - a^2}{2ar[r \tan \alpha - \sqrt{(r^2 - r^2)}]} \]

for the polar equation of the curve; where \( \rho \) is the radius vector and \( \theta \) the angle between the radius vector and a diameter through \( O \).

---

102. "Prove that in every triangle the square of the sum of the squares of the sides exceeds double the sum of their fourth powers."

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

Let \( a, b, c \) be the sides. Then \( a + b - c > 0, a + c - b > 0, b + c - a > 0, \) and \( a + b + c > 0 \). By multiplication, \( -a^4 - b^4 - c^4 + 2a^2b^2 + 2a^2c^2 + 2b^2c^2 > 0 \). Adding to both memb's \( 2(a^4 + b^4 + c^4) \), we have \( (a^2 + b^2 + c^2)^2 > 2(a^4 + b^4 + c^4) \).
103. “Through two given points draw a circle bisecting the circumference
of a given circle.”

SOLUTION BY HENRY HEATON, DESMOINES, IOWA.

Let $A$ and $B$ represent the given points and $GFE$ the given circle.
Through $A$ and $B$ describe any circle $ABEF$, intersecting the given circle in $E$ and $F$. Draw $EF$ and prolong it until it meets $AB$ produced in $H$.
Through the center $C$, draw $HG$ cutting the given circle in $D$ and $G$; then will $D$ and $G$
be points in the required circle. For $AH \times BH = FH \times EH = GH \times DH$; and if a
circle be passed through the points $A$, $B$ and $D$
it will cut $HG$, or $HG$ produced, at some point$I$; and we have $AH \times BH = IH \times DH = GH \times DH$. Hence $I$ coincides with $G$, and
the circle $ABD$ passes through $G$.

[The solutions by Dr. Eggers, Prof. Johnson, Prof. M. C. Stevens and
Mr. Seitz are substantially the same as the above. Prof. Stevens’ solution
was received too late to be included in our notice of solutions, published in
No. 2. We have, also, an elegant solution of this question by Miss Ladd
which should have been included with the notices in No. 2, but was omitted
through an oversight.]

104. “Show that $x^3 + 1 = y^3$ is possible for the values $x=0,-1$, and
1, only.”

SOLUTION BY PROF. A. B. EVANS, LOCKPORT, NEW YORK.

If $x^3 + 1 = y^3$ we may put $x = m - 1$; then

$$x^3 + 1 = m^3 - 3m^2 + 3m = \square.$$ \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \do
106. "Interpret \( \frac{a^2 - \beta a^3}{\sqrt{a^2}} \)."

**SOLUTION BY PROF. P. E. CHASE, PHILADELPHIA, PA.**

\[
\frac{\alpha \beta - \beta \alpha}{\sqrt{\alpha \beta}} = \frac{\alpha \beta (\beta - \alpha)}{\sqrt{\alpha \beta}} + \frac{\alpha \beta (\beta + \alpha)}{\sqrt{\alpha \beta}} = \frac{(\alpha - \beta) \cos \theta}{\epsilon \sin \theta} + (\alpha + \beta)
\]

is the diagonal of a parallelogram, of which one side is parallel to one diagonal of \((\alpha, \beta)\) and the other side is perpendicular to the other diagonal.

**SOLUTION BY CHRISTINE LADD, UNION SPRINGS, N. Y.**

If \( \rho \) be the radius of the circle which circumscribes the quaternion \( \frac{\beta}{\alpha} \), we have, by similar triangles, \( 2 \rho = \frac{\alpha - \beta}{\rho} \), in which \( \delta \) is the perpendicular from the extremity of \( \beta \) upon \( \alpha \). But \( \delta = - \frac{1}{\alpha} \sqrt{\alpha \beta} \); hence,

\[
2\rho = \frac{\alpha - \beta}{\alpha^{-1} \sqrt{\alpha \beta}} = \frac{\alpha^2 \beta - \beta^2 \alpha}{\sqrt{\alpha \beta}} = \frac{a^2 - \beta a^3}{\sqrt{a^2}}.
\]

If we put \( \gamma = \alpha - \beta \), \( 2\rho = \frac{\alpha \gamma \beta}{\sqrt{\beta \alpha}} = \frac{\gamma \beta a}{\sqrt{\alpha \gamma}} = \frac{\beta a \gamma}{\sqrt{\alpha \beta}} \).

99.—[As several of our correspondents object to the published solution of this problem, as being impracticable, we add the following:]

**SOLUTION BY PROF. M. C. STEVENS, SALEM, OHIO.**

The construction is sufficiently explained by the figure, in which \( AC = CD = DE = AB \) = radius of the circle, \( A \) and \( B \) being the two given points and \( G \) and \( H \) the required points; \( AF = EF = AD \) and \( AG = BF \).

Demonstration. Because \( AF^2 = AD^2 = 3AB^2 \)

\[= AB^2 + BH^2, \ldots \]

\[= 3AB^2 - AB^2 = 2AB^2, \ldots \]

\[\angle ABG \text{ is a right-angle.}\]
SOLUTION OF PROBLEMS IN NUMBER TWO.

Solutions of problems in No. 2 have been received as follows: From Marcus Baker, 107; G. M. Day, 107; Prof. A. B. Evans, 107, 108, 109 and 110; E. S. Farrow, 107; Henry Gunder, 107 and 109; H. Heaton, 107, 108 and 110; Prof. A. Hall, 109; Prof. Del. Kemper, 107 and 109; Christine Ladd, 107; Prof. H. T. J. Ludwick, 109; Artemas Martin, 107, 108 and 109; Dr. A. B. Nelson, 107 and 109; L. Regan, 107; E. B. Seitz, 107, 108 and 110.

We omitted, by oversight in No. 2, to credit Dr. Eggers with a solution of problem 102.

107. "If \( c = \sqrt[4]{a^2 - b^2} \), show that

\[
\frac{a + c}{a - c} = \left( \frac{c + a - b}{c - a + b} \right)^2.
\]

SOLUTION BY CHRISTINE LADD, UNION SPRINGS, N. Y.

If we put \( m = a + c, n = a - c \), the equation becomes

\[
\frac{m}{n} = \left( \frac{m - b}{b - n} \right)^2.
\]

Hence

\[
m(b^2 - 2bn + n^2) = n(m - 2mb + b^2),
\]

\[
b^2 = \frac{nm^2 - mn^2}{m - n} = \frac{mn}{m - n} = a^2 - c^2, \text{ whence } c = \sqrt[4]{a^2 - b^2}.
\]

108. "Employing the notation of Prob. 100, page 31, show by Finite Differences, or otherwise, that

\[
\frac{1}{a_z} + \frac{1}{b_z} = \frac{1}{c_z} = \frac{1}{6} \left( \left[ 5 + 2\sqrt{6} \right]^z + \left[ 5 - 2\sqrt{6} \right]^z + 2 \right) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) + \frac{1}{2\sqrt{6}} \left( \left[ 5 + 2\sqrt{6} \right]^z - \left[ 5 - 2\sqrt{6} \right]^z \right) \left( \sqrt{\frac{2}{ac} + \frac{2}{ab} + \frac{2}{bc}} \right); \text{ where the double sign is to be taken plus when } x \text{ is odd and minus when } x \text{ is even.}
\]

SOLUTION BY PROF. A. B. EVANS, LOCKPORT, N. Y.

Let

\[
\frac{1}{a_z} + \frac{1}{b_z} + \frac{1}{c_z} = u_z \text{ and } \left( \frac{2}{a_z b_z} + \frac{2}{a_z c_z} + \frac{2}{b_z c_z} \right)^{\frac{3}{2}} = v_z; \ldots (1)
\]

then

\[
\frac{1}{a_z} = -\frac{1}{a_{z-1}} + 2u_{z-1} + 2v_{z-1}, \quad \frac{1}{b_z} = -\frac{1}{b_{z-1}} + 2u_{z-1} + 2v_{z-1},
\]

\[
\frac{1}{c_z} = -\frac{1}{c_{z-1}} + 2u_{z-1} + 2v_{z-1}. \ldots \ldots (2)
\]
From (1) when \( x = 0, \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = u \) and \( \left( \frac{2}{ab} + \frac{2}{ac} + \frac{2}{bc} \right)^{\frac{1}{2}} = v; \ldots \) (3)

and from (2) when \( x = 1, \)
\[
\frac{1}{a_1} = \frac{1}{a} + 2u + 2v, \quad \frac{1}{b_1} = \frac{1}{b} + 2u + 2v, \quad \frac{1}{c_1} = \frac{1}{c} + 2u + 2v \ldots (4)
\]

From (4),
\[
\frac{1}{a_1} + \frac{1}{b_1} + \frac{1}{c_1} = u_1 = 5u + 6v, \ldots \ldots \ldots \ldots (5)
\]

and
\[
\left( \frac{2}{a_1 b_1} + \frac{2}{a_1 c_1} + \frac{2}{b_1 c_1} \right)^{\frac{1}{2}} = v_1 = 4u + 5v, \ldots \ldots \ldots \ldots (6)
\]

From (2), \( u_x = 5u_{x-1} + 6v_{x-1} \) and \( v_x = 4u_{x-1} + 5v_{x-1} \ldots \ldots \ldots (7) \)

By giving \( x \) integral values from 1 to \( x \) inclusive, we may obtain a series of equations from (2) that will enable us to find readily
\[
\frac{1}{a_x} \pm \frac{1}{b_x} = \frac{1}{c_x} \pm \frac{1}{c} = 2[(u_{x-1} + v_{x-1}) - (u_{x-2} + v_{x-2}) + (u_{x-3} + v_{x-3}) \ldots \pm (u + v)]; \ldots (8)
\]

where the double sign is to be taken \( + \) when \( x \) is odd and \( - \) when \( x \) is even.

From the first of equations (7) \( v_{x-1} = \frac{1}{6}(u_x - 5u_{x-1}), \) and therefore \( v_x = \frac{1}{5}(u_{x+1} - 5u_x), \) These values of \( v_{x+1} \) and \( v_x, \) substituted in the second of equations (7), give \( u_{x+1} - 10u_x + u_{x-1} = 0. \ldots u_x - 10u_{x-1} + u_{x-2} = 0. \) (9)

Equation (9) is an equation in finite differences whose solution is
\[
u_x = C_1 r_1^x + C_2 r_2^x; \ldots \ldots \ldots \ldots (10)
\]

where \( r_1 = 5 + 2\sqrt{6} \) and \( r_2 = 5 - 2\sqrt{6} \) are the roots of \( x^2 - 10x + 1 = 0, \) and \( C_1, C_2 \) are constants of integration. To determine these constants, we observe that when \( x = 0 \) and \( x = 1, \) equation (10) gives \( u = C_1 + C_2 \) and \( u_1 = 5u + 6v = C_1 r_1 + C_2 r_2 \); and therefore \( C_1 = \frac{u}{r_1} + \frac{v}{r_2} \) and \( \)
\[
C_2 = \frac{1}{r_2} u - \frac{1}{r_1} v / 6. \ldots \ldots \ldots \ldots \ldots (11)
\]

Again, since \( v_x = \frac{1}{5}(u_{x+1} - 5u_x) \), we have
\[
u_x + v_x = \frac{1}{5}(u_{x+1} + u_x) = \frac{1}{5} C_1 (r_1^{x+1} + r_1^x) + \frac{1}{5} C_2 (r_2^{x+1} + r_2^x). \ldots \ldots \ldots \ldots \ldots (12)
\]

Observing that \( r_1 + 1 = 6 + 2\sqrt{6} \) and \( r_2 + 1 = 6 - 2\sqrt{6} \) we may write
\[
u_x + v_x = \frac{1}{5}(6 + 2\sqrt{6}) C_1 (r_1^x) + \frac{1}{5}(6 - 2\sqrt{6}) C_2 (r_2^x). \ldots \ldots \ldots \ldots \ldots (13)
\]

From (8) and (13) we find
\[
\frac{1}{a_x} \pm \frac{1}{b_x} = \frac{1}{c_x} \pm \frac{1}{c} = \frac{1}{a} \pm \frac{1}{b} \pm \frac{1}{c} \pm \frac{1}{c} = \frac{1}{a} \pm \frac{1}{b} \pm \frac{1}{c}
\]

\[
\frac{1}{a} \pm \frac{1}{b} \pm \frac{1}{c} = \frac{1}{a} \pm \frac{1}{b} \pm \frac{1}{c} = \frac{1}{a} \pm \frac{1}{b} \pm \frac{1}{c} = \frac{1}{a} \pm \frac{1}{b} \pm \frac{1}{c}
\]

\[
\frac{u}{5} \left[ (5 + 2\sqrt{6})^x + (5 - 2\sqrt{6})^x \pm 2 \right] + \frac{v}{2\sqrt{6}} \left[ (5 + 2\sqrt{6})^x - (5 - 2\sqrt{6})^x \right]
\]

\[
\frac{1}{5} \left[ (5 + 2\sqrt{6})^x + (5 - 2\sqrt{6})^x \pm 2 \right] \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right)
\]
109. "Show that the determinant
\[
\begin{vmatrix}
  a & b & c & d \\
  b & a & d & c \\
  c & d & a & b \\
  d & c & b & a \\
\end{vmatrix}
\]
is divisible by \((a + b)^4 - (c + d)^4\); and by \((a - b)^4 - (c - d)^4\)."

**SOLUTION BY HENRY GUNDER, NORTH MANCHESTER, IN.**

Adding the 2nd, 3rd and 4th columns to the first, we have
\[
\begin{vmatrix}
  1 & b & c & d \\
  1 & a & d & c \\
  1 & d & a & b \\
  1 & c & b & a \\
\end{vmatrix}
\]

\((a+b+c+d)\)

Subtracting the 3rd and 4th, and adding the 2nd column to the 1st, we get
\[
\begin{vmatrix}
  1 & b & c & d \\
  1 & a & d & c \\
  -1 & d & a & b \\
  -1 & c & b & a \\
\end{vmatrix}
\]

\((a+b-c-d)\)

This shows that the given determinant is divisible by \((a+b+c+d)\) and by \((a+b-c-d)\), and hence by their product \((a+b)^4 - (a + d)^4\).

By adding the 3rd column and subtracting the 2nd and 4th, we get the factor \((a - b + c - d)\). Subtracting the 2nd and 3rd and adding the 4th column we get the factor \((a - b - c + d)\). Therefore the determinant is divisible by \((a - b)^4 - (c - d)^4\). (See Analyst, Vol. III, No. 1.)

[This question was solved in a manner similar to the above by D. J. McAdam, Dr. Nelson, Prof. Kemper and Prof. Hall.]

**SOLUTION BY ARTEMAS MARTIN, ERIE, PA.**

By reduction the value of the determinant is found to be
\[
(a^4 + b^4) - (a-b)^4(c+d)^4 - (a+b)^4(c-d)^4 + (c^2 - d^2)^4,
\]

\[= [(a + b)^4 - (c + d)^4][(a - b)^4 - (c - d)^4].\]

[Prof. Ludwick and Prof. Evans solved this question in the same manner as Mr. Martin.]

110. "A circle, radius \(r\), is placed at random on another equal circle. Prove that the average area of the greatest ellipse that can be inscribed in the area common to the two circles, is \(\pi r^2 \left(\frac{4}{\pi} - \frac{3}{4}\right)\)."

**SOLUTION BY HENRY HEATON, DES MOINES, IOWA.**

Let \(2a\) = the distance between the centers of the circles. The distance between the center of the ellipse and that of one of the circles will = \(a\), and the equation of the ellipse, taking the center of one of the circles as
the origin, is
\[ B^2(x - a)^2 + A^2y^2 = A^2B^2; \]
whence
\[ \frac{dy}{dx} = -\frac{B^2(x - a)}{A^3y}. \]

The equation of the circle is \( x^2 + y^2 = r^2. \)

\[ \frac{dy}{dx} = -\frac{x}{y} \]

At the point where the curves are tangent, \( x, y, \) and \( dy/dx \) of the ellipse and \( x, y, \) and \( dy/dx \) of the circle coincide. Combining the above equations we get
\[ A = \sqrt{\left[ (r^2 - ax)\left( \frac{x-a}{x} \right) \right]}, \]
and \( B = \sqrt{(r^2 - ax)}. \)

\[ A = (r^2 - ax)\sqrt{\left( \frac{x-a}{x} \right)}. \]

\[ \frac{dA}{dx} = \frac{a}{2x^3} (r^2 - ax) \sqrt{\left( \frac{x}{x-a} \right)} \]

Putting this differential coefficient = 0 to find the value of \( x \) that will make \( A \) a maximum, we get \( x = \frac{1}{a} \sqrt{(8r^2 + a^2)} + a, \) or
\[ a = \frac{2x^2 - r^2}{x}. \]

\[ da = \left( \frac{2x^2 + r^2}{x^2} \right) dx. \]

The mean area of the ellipse
\[ = \frac{1}{\pi r^2} \int_0^{2\pi} \pi AB \times 2\pi ada = \frac{2\pi}{r} \int_0^{2\pi} AB \times ada \]

\[ = \frac{4\pi}{r^2} \int_{4r/2}^{r} \frac{4x - r^2}{x^4} \times (r^2 - x^2)^{1/2} dx = \frac{16\pi}{r^2} \int_0^{4\pi/3} (r^2 - x^2)^{1/2} dx \]

\[ - 4\pi r^2 \int x^4 (r^2 - x^2)^{1/2} dx = \pi r^2 \left( \frac{\pi}{2} - \frac{4}{3} \right). \]

**SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.**

Let \( A \) and \( B \) be the centers of the two circles, \( CD \) the common chord, \( OM \) and \( ON \) the semi-axes of the greatest ellipse that can be inscribed in the area common to the two circles.

Put \( AC = r, AB = z, OM = x, ON = y. \)

Then if we represent the coordinates by \( m, n, \) the equation to the circle whose center is \( A, \) referred to \( AB \) and \( CD, \) is \( (m + \frac{1}{2}z)^2 + n^2 = r^2. \)

The eqn. to the ellipse is \( x^2m^2 + y^2m^2 = x^2y^2. \)

From (1) and (2) by eliminating \( n, \) we have
\[ (x^2 - y^2)m^2 - y^2zm + y^2(r^2 - x^2 - \frac{1}{2}z^2) = 0. \]

But, since the ellipse is tangent to the circle, the two values of \( m \) in (3) must be equal; hence we have
\[ (x^2 - y^2)(4r^2 - 4x^2 - z^2) = y^2z^2. \]
From (4) we find \( x^2y^2 = x^4 - \frac{x^4z^2}{4(r^2 - x^2)} \) a maximum. . . . . (5)

Differentiating (5), regarding \( z \) as constant, equating to zero, and reducing, we have

\[
(2r^2 - x^2)z^2 = 8(r^2 - x^2)^3. \quad \ldots \ldots \ldots \quad (6)
\]

From (6) we have

\[
z^2 = \frac{8(r^2 - x^2)^3}{2r^2 - x^2}. \quad \ldots \ldots \ldots \quad (7)
\]

Substituting the value of \( z^2 \) in (5), reducing, and extracting the square root, we have

\[
xy = \frac{x^8}{(2r^2 - x^2)^2}. \quad \ldots \ldots \ldots \quad (8)
\]

Differentiating (7), we have

\[
dx = \frac{84x^2dx}{(2r^2 - x^2)^2} - 8x^2. \quad \ldots \ldots \ldots \quad (9)
\]

If \( A \) denotes the average area of the ellipse, we have

\[
A = \frac{\int_0^{2\pi} xxy.2\pi r dx}{\int_0^{2\pi} 2\pi r dx} = \frac{\pi}{2r^2}\int_0^{2\pi} xydx. \quad \ldots \ldots \quad (10)
\]

Substituting in (10) from (8) and (9), and observing that when \( z = 0, x = r \), and when \( z = 2r, x = 0 \), we have

\[
A = \frac{4\pi r^2}{r^2}\int_0^{2\pi} \frac{x^4 dx}{(2r^2 - x^2)^2} - 4\pi r^2 \int_0^{2\pi} \frac{x^4 dx}{(2r^2 - x^2)^2}. \quad \ldots \ldots \quad (11)
\]

Putting \( \sin \theta = x + r \sqrt{2} \), (11) becomes

\[
A = 16\pi r^2 \int_0^{\pi/2} \sin^4 \theta d\theta - 4\pi r^2 \int_0^{\pi/2} \tan^4 \theta d\theta = \pi r^2 \left( \frac{\pi}{2} - \frac{\pi}{2} \right). \quad \ldots \ldots \ldots \quad (12)
\]

[PROB. 89. Though we have already published two solutions of this problem, yet as both are in some respects confessedly defective, and as the method adopted in the following solution differs from both the published solutions, we add another solution of 89.]

In justice to Mr. Siverly and to Mr. Adcock, we take pleasure in stating that Mr. Siverly corrected his statement that the spheres would roll in opposite directions, several months ago, and by rigorous analysis had arrived at the same conclusion, in relation to the rolling of the lower sphere, as that deduced in the following solution by Mr. Hill. And, since the issue of No. 2, we have received from Mr. Adcock a revision of his published solution, in which, after correcting equations (6) and (9), he obtains an equation for the required curve which corresponds with that given by Mr. Siverly, and consequently with the following by Mr. Hill. And he also arrives at the same conclusion as Mr. Siverly and Mr. Hill, in relation to the rolling of the lower sphere.
SOLUTION OF PROBLEM 89. (SEE PAGE 195, VOL. II.)

BY G. W. HILL.

Let \( x \) and \( 0 \) be the coordinates of the lower sphere, \( x' \) and \( y' \) those of the upper, \( \theta \) and \( \theta' \) the amounts of rotation, and \( \varphi \) the angle the line joining their centres makes with the horizon, and for brevity put \( h = R + r \).

The expression for the living force is

\[
T = \frac{m}{2} \left[ \frac{d^2x}{dt^2} + \frac{2}{5} R^2 \frac{d\theta^2}{dt^2} \right] + \frac{m'}{2} \left[ \frac{d^2x'}{dt^2} + \frac{dy'^2}{dt^2} + \frac{2}{5} r^2 \frac{d\theta'^2}{dt^2} \right],
\]

and the potential is \( \Omega = -m'y' \).

According to the frictional conditions, the variables \( x, x', y', \theta, \theta' \) satisfy the following equations,

\[
\begin{align*}
R\theta - x &= 0, \\
r\theta' + x + h \tan^{-1} \frac{y'}{x'-x} &= 0, \\
\sqrt{[(x' - x)^2 + y'^2]} - h &= 0. \\
\end{align*}
\]

(1)

With Lagrange's method of multipliers, if we denote these equations respectively by \( L = 0, M = 0, N = 0 \), and the multipliers of their differentials by \( \lambda, \mu, \nu \), and take \( \xi \) to represent any one of the 5 variables \( x, x', y', \theta, \theta' \), the general differential equation of the problem is

\[
\frac{d^2T}{dt^2} \frac{d^2T}{d\xi^2} - \frac{dT}{d\xi} \frac{dT}{d\xi} = \frac{d\Omega}{d\xi} + \lambda \frac{dL}{d\xi} + \mu \frac{dM}{d\xi} + \nu \frac{dN}{d\xi}.
\]

Applying this in succession to each of the 5 variables and writing for simplicity \( \varphi \) for \( \tan^{-1} \frac{y'}{x'-x} \), we get

\[
\begin{align*}
m \frac{d^2x}{dt^2} &= -\lambda + \mu [1 + \sin \varphi] - \nu \cos \varphi, \\
m \frac{d^2x'}{dt^2} &= -\mu \sin \varphi + \nu \cos \varphi, \\
m \frac{d^2y'}{dt^2} &= -m'g + \mu \cos \varphi + \nu \sin \varphi, \\
\frac{2}{5} m R^2 \frac{d^2\theta}{dt^2} &= \lambda R, \\
\frac{2}{5} m r^2 \frac{d^2\theta'}{dt^2} &= \mu r.
\end{align*}
\]

(2)

Adding the first and second of (2)

\[
\frac{d^2(mx + m'x')}{d\xi^2} = \mu - \lambda.
\]
The two first of (1) and the two last of (2) give
\[ \lambda = \frac{2}{5} \frac{mR}{\alpha^2} \frac{d^2 \theta}{d\alpha^2} = \frac{2}{5} \frac{m}{\alpha^2} \frac{d^2 x}{d\alpha^2}, \]
\[ \mu = \frac{2}{5} m \frac{dr}{d\alpha} \frac{d^2 \theta'}{d\alpha^2} = -\frac{2}{5} m' \left[ \frac{d^2 x}{d\alpha^2} + \frac{h}{\alpha^2} \frac{d^2 \phi}{d\alpha^2} \right]. \]
Substituting these values for \( \lambda \) and \( \mu \) in the last equation,
\[ \frac{d^2 (mx + m'x')}{d\alpha^2} = -\frac{2}{5} \left( m + m' \right) \frac{d^2 x}{d\alpha^2} - \frac{2}{5} m' h \frac{d^2 \phi}{d\alpha^2}. \ldots \ldots \ \ (3) \]
Integrating once and eliminating \( x' \)
\[ \frac{7}{5} \left( m + m' \right) \frac{dx}{dt} + m' h \left( \frac{2}{5} - \sin \varphi \right) \frac{d\varphi}{dt} = 0, \]
where the constant is zero because the spheres are supposed to set out together from a state of rest. As \( d\varphi / dt \), in general, is negative, \( \varphi \) can always be supposed in the first quadrant, it is evident from this equation, that if \( \sin \varphi > \frac{2}{5} \), the lower sphere will move horizontally to the side on which the upper sphere is; but if \( \sin \varphi < \frac{2}{5} \), in the opposite direction.
Integrating (3) twice
\[ (7m + 2m')x + 5m'x' + 2m'h \varphi = \text{a constant.} \]
Eliminating \( x \) and \( \varphi \) from this by substituting their values in terms of \( x' \) and \( y' \), we get as the equation of the path of the centre of the upper sphere
\[ 7 \left( m + m' \right) \left[ x' - \sqrt{(h^2 - y'^2)} \right] + m' \left[ 2h \sin^{-1} \frac{y'}{h} + 5 \sqrt{(h^2 - y'^2)} \right] = \text{a constant.} \]
As \( \nu \) denotes the pressure of the upper on lower sphere, the spheres will separate when \( \nu = 0 \). Now if we eliminate \( \mu \) between the second and third of (2), we see that \( \nu = 0 \) is equivalent to
\[ \frac{d^2 x'}{d\alpha^2} \cos \varphi + \left( \frac{d^2 y'}{d\alpha^2} + g \right) \sin \varphi = 0. \]
And if we eliminate \( x' \) and \( y' \) from this by means of their values in terms of \( x \) and \( \varphi \) we get
\[ \frac{d^2 x}{d\alpha^2} \cos \varphi + g \sin \varphi - h \frac{d^2 \varphi}{d\alpha^2} = 0. \]
By eliminating second derivatives this becomes
\[ 49m \left[ \frac{h}{g} \frac{d^2 \varphi}{d\alpha^2} - \sin \varphi \right] + 10m'(1 + \sin \varphi)^2 \left[ \frac{h}{g} \frac{d^2 \varphi}{d\alpha^2} - 1 \right] = 0, \]
which, by substituting the value of \( d\varphi^2 / d\alpha^2 \), becomes
\[ 70(m + m') \left[ 49m + 10m' + 20m' \sin \varphi + 10m' \sin^2 \varphi \right] \left[ \sin \beta - \sin \varphi \right] - \left[ 10m' + (49m + 20m') \sin \varphi + 10m' \sin^2 \varphi \right] \left[ 49m + 45m' + 20m' \sin \varphi - 25m' \sin^2 \varphi \right] = 0. \]
NOTE BY PROF. W. W. JOHNSON.—Problem 85, in Analyt for September, is identical with No. 85 of the Exercises in Chauvenet’s Geometry, the statement being made in the form which Prof. Scheffer gives in the November No. Prof. Scheffer is undoubtedly right in stating that the problem cannot be solved by the rule and compass only. The following will show that it has theoretically six solutions, and in some cases they are all real solutions.

Assume the given point as origin and the line joining it with the centre of one of the circles as initial line, the polar equation of that circle is

\[ r^2 - 2ar \cos \theta + a^2 = b^2, \]

or

\[ r = a \cos \theta \pm \sqrt{(b^2 - a^2 \sin^2 \theta)}. \]

Construct a curve by increasing and diminishing each radius vector by \( d \), the length of the fourth side of the quadrilateral or line to be placed between the circumferences. The equation to this curve is therefore

\[ r = a \cos \theta \pm d \pm \sqrt{(b^2 - a^2 \sin^2 \theta)}, \]

or

\[ r^2 - 2ar \cos \theta = 2d(r - a \cos \theta) + a^2 + d^2 - b^2 = 0, \]

or, partially introducing rectangular coordinates,

\[ x^2 + y^2 - 2ax + a^2 + d^2 - b^2 = \pm 2d(r - a \cos \theta). \]

Multiplying both members by \( r \) and squaring we have

\[ (x^2 + y^2)(x^2 + y^2 - 2ax + a^2 + d^2 - b^2)^2 = 4d^2(x^2 + y^2 - ax)^2. \]  

Now whenever this locus is cut by the other circle we have a solution of the problem. The equation of this last circle will be of the form

\[ x^2 + y^2 + Ax + By + C = 0. \]

Substituting the value of \( x^2 + y^2 \) from (2) into (1), (namely the linear expression, \(-Ax - By - C\)) we have an equation of the third degree; which cubic combined with equation (2) will give theoretically six values of \( x \) and \( y \). Constructing the curve (1) in the case when the distance of the pole from the circumference of the first circle is less than \( d \), it is evident that the circle (2) may cut (1) in six real points, hence there may be six real solutions. The position of the given circles and the given point in this case is as follows — The circles being of unequal radii intersect, and the given point is within the smaller of the crescent-shaped figures formed. Let a straight line pass through the given point and revolve about it, the intercepts made on this line by each of the crescent-shaped and the lens-shaped figures will vary, and each may evidently twice become equal to the line to be inscribed provided the latter is within the proper limits.
PROBLEMS.

111. By Prof. J. H. Kershner, Mercersburg, Pa. — Given
\[ \frac{x + y}{x - y} + \frac{x - y}{x + y} = 3\frac{1}{2} \ldots (1), \quad x^2 + y^2 = 45, \ldots \ldots (2) \]
to be solved without the use of an auxiliary unknown.

112. By Prof. M. L. Comstock, Galesburg, Ill. — Wishing to know the height of a tower standing at the summit of a slope on the opposite side of the street, and not being able to leave my room, I measured the angles of elevation of the bottom and top of the tower 40°, 70°, respectively, and the angle of depression of the foot of the slope 40°; and a passer-by carried a tape-line across, giving me the distance from my point of observation to the foot of the slope, 50 feet. I knew the angle made by the face of the slope with the horizontal plane of the street to be 60°. From these data, I found the height of the tower, having given, \( \log \tan 20° = 9.651066 \), \( \log \sin 50° = 9.884254 \), \( \log 10 = 1 \), and \( \log 171072 = 5.233188 \).

113. By Marcus Baker, U. S. C. S., Washington, D. C. — The sides of a plane triangle are in arithmetical progression, common difference \( d \), and the angle opposite the least side is one third of the angle opposite the greatest side; construct the triangle.

114. By Dr. Nelson. — Required the shortest proof of the Pons Asinorum. (Eucl. 47, 1.)

115. By the Editor. — Show, by a geometrical construction, that the results obtained by Miss Ladd and Prof. Chase, in their solutions of Prob. 106, are the same.

116. By Artemas Martin, Erie, Pa. — A sector less than a semicircle is cut at random from a given circle, and a circle inscribed in it. Find the average area of this inscribed circle.

117. By F. P. Matz, B. E., Kutztown, Pa. — Required the average area of all the acute-angled triangles that can be inscribed in a given ellipse.

118. By Prof. Johnson. — Find the general relation which exists between the four sides and the two diagonals of any quadrilateral. Consider particularly the case when the opposite sides are equal.

Query 1. By T. P. Stowell, Rochester, N. Y. — Which is the most effective; a break applied at the top or the side of a car wheel in motion?

Query 2. By Cadet E. S. Farrow, West Point, N. Y. — Can the equation \( x^2 + \sqrt{x} = a \), be solved?
NOTE. I was not aware of the utility, until I saw Mr. Evans' rule, of any rule for the extraction of roots above the common one. I have derived his formula, and also the following which will give the value of \( R \) to one-half more decimal places than Mr. Evans' rule:

\[
R = r \times \frac{(n + 1)N + (n - 1)r^n}{(n - 1)N + (n + 1)r^n},
\]

where the letters represent the same numbers as in the formula given by Mr. Evans, *Analyt*, page 11, Vol. III.

*Example.* To find the square root of 2. Here \( n = 2, N = 2 \), and let the approximate trial root be 1.4, then

\[
R = 1.4 \times \frac{3 \times 2 + (1.4)^3}{2 + 3(1.4)^3} = 1.4 \times \frac{7.96}{7.88} = 1.4 \times \frac{199}{197} = 1.4142133
\]

correct to 6 places of decimals. In this formula, if \( r \) be too great \( R \) will also, and vice versa, and therefore by using two trial roots, one too great and the other too small, the limits, between which the true root is, are obtained.

R. J. Adcock.

[98. Mr. Adcock objects to the published solution of 98. He finds the height due the velocity of the stream (13.2 feet per sec.) to be 2.7+ feet. And because the pressure of a fluid column of that height would produce, in an equal sectional area of the stream, the given velocity, he contends that this pressure, viz., \( 2.7+ \times 62\frac{1}{2} = 169.2 \) is the correct answer.]

---

**BOOK NOTICE.**

*Elements of Geometry* With Exercises for Students, and an Introduction to Modern Geometry, By A. Schuyler, LL. D., President of Baldwin University, &c.

This book will interest students on account of its superior mechanical execution; but it is especially valuable for the logical accuracy with which the various propositions are announced and demonstrated. The Introduction to Modern Geometry is, for many students, a valuable addition.

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**ERRATA.**

On page 35, erase lines 9, 10 and 11.

" " 41, " third member in lines 9, 10 and 11.

" " 53, lines 18, 20 and 22, for \( X(a_1 \) or \( b_1 \)) read \( X_1(a_1 \) or \( b_1 \).

" " 63, last line, for constant angle at \( N, \&c, \) read constant length of \( NC. \)

" " 75, line 5 from bottom for \( \frac{cc}{bh} \), read \( \frac{cc}{hh} \).
THE ANALYST.

Vol. III. July, 1876. No. 4.

PROBLEM. — TO FIT TOGETHER TWO OR MORE QUADRICs SO THAT THEIR INTERSECTIONS SHALL BE PLANE.

BY PROF. E. W. HYDE, CINCINNATI, OHIO.

Let us first determine a condition that the intersection may be projected upon some given plane into a conic. This will evidently be the case whenever between the equations of the two quadrics we can eliminate one of the variables, say \( z \), without raising the degree of the resulting equation above the second. Thus, if the equations are

\[
A_1x^2 + B_1xy + C_1y^2 + D_1z^2 + E_1x + F_1y + G_1 = 0,
\]

\[
A_2x^2 + B_2xy + C_2y^2 + D_2z^2 + E_2x + F_2y + G_2 = 0,
\]

it is plain that after elimination of \( z \) the resulting equation will be the general equation of the 2nd degree between \( x \) and \( y \). If we regard the given equations as referred to oblique coordinates, each represents a quadric having its center in the plane \( xy \) and the axis of \( z \) parallel to the diameter conjugate to \( xy \). The same is true if the axes of reference be rectangular, but in this case the plane \( xy \) is a plane of symmetry, and \( z \) is parallel to an axis of symmetry of each surface.

Hence if two quadrics have a common plane of symmetry the orthogonal projection of their intersection upon this plane is a conic.

Evidently equations of the second degree in \( x, y \) and \( z \) can be presented in a variety of forms such that \( z \) may be eliminated without giving an equation in \( x \) and \( y \) of higher degree than the second, but that given is sufficient for our purpose.

Suppose now that we have two quadrics whose equations are those given above, and which are so related in dimensions and position that one pierces entirely through the other, cutting from it two separate curves. These being projected on the plane of \( xy \) in a conic, the latter must be a
hyperbola, since no other conic has two separate branches. Now let the smaller quadric increase in size, then the two branches of the hyperbola will approach each other, i.e. their transverse axis will become shorter, and when the two quadrics exactly cut each other off the hyperbola will be reduced to a pair of intersecting right lines, that is the curves of intersection will be plane. The points in which these two plane curves intersect will be at the extremities of a chord common to the two quadrics, and at these two points common tangent planes to the two surfaces may be drawn.

*Hence the intersection of two quadrics will be plane, whenever they have a common chord, at whose extremities planes may be drawn tangent simultaneously to the two surfaces.*

If the two quadrics have a common plane of symmetry the common chord will be perpendicular to this plane.

In the figure let $MABCDF$ represent an ellipsoid having one plane of symmetry coincident with the paper, and the axis perpendicular to this plane of any length, and let $GKH$ be another ellipsoid inscribed within the first, touching it along the ellipse $GOH$ projected upon the paper in a right line.

(It may be shown that the curve of contact of two quadrics of which one is inscribed within the other is always plane.) Let some other quadric, as the paraboloid $ALEDC$, be described about the ellipsoid $GKHL$ touching it in the ellipse $KOL$ projected into a right line on the paper. If planes be drawn at the two points projected in $O$, i.e. at the intersections of the curves $GOH$ and $KOL$, tangent to the ellipsoid $GKHL$, they will also be tangent at the same points to the larger ellipsoid and the paraboloid; hence these must intersect each other in plane curves projected into the right lines $AOD$ and $COE$.

Hence for the solution of the problem stated at the beginning of this article we have only to inscribe within one of the given quadrics some other quadric of such a nature that the second given surface may be circumscribed about it, then will the intersections of the two given surfaces be plane.
If the given surfaces are of revolution, as is most frequently the case in practice, the surface to be inscribed is a sphere, and the construction is very simple.

If we now consider the curves in the figure without reference to the surfaces represented by them we may arrive at some interesting properties of conics.

If a conic $GKLH$ be inscribed within any other conic $MBCDF$, and about the first some third conic $CDLA$ be described, the locus of the intersection $O$ of the lines $AD$ and $CE$ joining the opposite points of the intersection of $CDLF$ and $MBCDF$ is the chord of contact $GH$ of the first and second conics. The truth of this is evident from what has preceded.

As a particular case let the third conic be a pair of tangents to $GKLH$; then we see that if pairs of tangents be drawn to $GKLH$, and their opposite points of intersection with $MACHF$ be joined by straight lines, the locus of the intersections of these lines is the chord of contact of $GKLH$ and $MACHF$. If the tangents touch $GKLH$ on opposite sides of $GH$, as in the case of $BF$ and $QR$, the point $P_1$ given by them is inside the ellipse, but if they touch $GKLH$ on the same side of $GH$, as $BF$ and $MN$, the point $P_2$ given by them is outside the ellipse.

We evidently have from this an easy construction for finding the points of contact of one conic inscribed within another. It is only necessary to draw any three tangents as $MN$, $BF$ and $QR$ to the inscribed conic, and join the opposite points $M$ and $F$, $B$ and $E$, $B$ and $R$, and $Q$ and $F$, obtaining thus points $P_1$ and $P_2$ of the chord of contact $GH$.

The chord of contact $GH$ is parallel to the diameters of $GKLH$ and $MBCDF$ which are conjugate to the common diameter $C_1C_2$ joining the centers of the two curves.

This is evidently true when $GKLH$ is a circle, $GH$ being then perpendicular to $C_1C_2$ which is an axis of symmetry of $MBCDF$. The common tangents to the two curves at $G$ and $H$ then meet on $C_1C_2$. Take any point not in the plane of the curves, as the vertex of a projecting cone, and project the figure upon some plane such that $GKLH$ shall become a circle $G'K'L'H'$. Then, as we have just seen, $G'H'$ will be conjugate in direction to $C_1C_2$, and the common tangents at $G'$ and $H'$ will meet on $C_1C_2$. Therefore, since tangency is a projective property, the tangents at $G$ and $H$ must meet on $C_1C_2$, and hence $GH$ is conjugate in direction to $C_1C_2$. If the circumscribing curve be a parabola as $AEDC$, the chord of contact $LK$ is parallel to the tangent at the extremity of the diameter of the parabola which passes through $C_2$, there being of course no conjugate diameter in this case.
CALCULATION OF RADICALS.

BY DR. H. EGERS, MILWAUKEE, WISCONSIN.

For want of room I shall confine myself to the statement of theorems and rules.

1. Theorem: Let \( z \) and \( x \) be any positive numbers and \( k \) any positive integer; further let \( x = \sqrt[k]{z} \). Form the expression

\[
P = x^{k-1} + x^{k-2}x + x^{k-3}x^2 + \ldots + ax^{k-2} + x^{k-1}
\]

\[
= \frac{x^k - z}{x - x^k};
\]

develop the power \( P^k \) in ascending powers of \( x \), and always substitute in this development \( x \) for \( x^k \), then the expression for \( P^k \) will assume the form

\[
P^k = A_k, s_{-1} + A_k, s_{-2}x + A_k, s_{-3}x^2 + \ldots + A_k, s_{-1}x^{k-1}. \ldots \ldots \ldots (1)
\]

Now the \( n \) successive ratios

\[
\frac{A_k, s_{-1}}{A_k, s_{-2}}, \frac{A_k, s_{-2}}{A_k, s_{-3}}, \ldots, \frac{A_k, s_{-1}}{A_k, s_{-1}}
\]

are \( n \) different expressions for the real root of \( n^{th} \) degree of \( s \) with the same degree of approximation; or what amounts to the same, for sup. lim. \( k = \infty \)

\[
\frac{A_k, s_{-1}}{A_k, s_{-1}} = \sqrt[k]{z}, \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (2)
\]

and superior limit \( k = \infty \) \[\frac{A_k, s_{-1}}{A_k, s_{-1}} = \sqrt[k]{x^s}.
\]

The calculated values of the quantities \( A \), which may be called components, are as follows:

Let \( A_0, A_1, A_2, \ldots \ldots \) be the binomial coefficients in the expansion of \( (1 + x)^k \), and \( B_0, B_1, B_2, \ldots \ldots \) the positive binomial coefficients in the expansion of \( (1 + x)^{-k} \), that is

\[
A_0 = 1, \quad A_1 = \frac{k}{1}, \quad A_2 = \frac{k(k-1)}{1.2}, \text{ etc.},
\]

\[
B_0 = 1, \quad B_1 = \frac{k}{1}, \quad B_2 = \frac{k(k+1)}{1.2}, \text{ etc.};
\]

then the calculated values of the components are

\[
A_k, s_{-1} = a^{k(s-1)} + (A_0B_n - A_1B_0)a^{k(s-1)-s} + (A_0B_2 - A_1B_1 + A_2B_0)a^{k(s-1)-2s} + \ldots
\]

\[
\times a^{k(s-1)-2s}x^2 + (A_0B_3 - A_1B_2 + A_2B_1 - A_3B_0)a^{k(s-1)-3s}x^3 + \ldots
\]

The series for this component as well as for all others is finite, for the expansion of \( P^k \) shows that the highest power of \( x \) must be smaller than \( k \).
The next component is

\[ A_{k, n-3} = B_k a^{n-1} - (A_0 B_{n+1} - A_1 B_1) a^{n-2} z + \ldots + (A_0 B_{n+1} - A_1 B_1) a^{n-2} z^3 + \ldots \]

\[ A_{k, n-3} = B_1 a^{n-1} - (A_0 B_{n+1} - A_1 B_1) a^{n-2} z + \ldots + (A_0 B_{n+1} - A_1 B_1) a^{n-2} z^3 + \ldots \]

where \( h \) denotes any of the numbers 0, 1, 2, \ldots, \( n - 1 \); and finally the last two components, and the most simple, are;

\[ A_{k, 1} = B_{n-2} a^{n-1} - (A_0 B_{n-2} - A_1 B_{n-2}) a^{n-2} z + \ldots + (A_0 B_{n-2} - A_1 B_{n-2}) a^{n-2} z^3 + \ldots \]

\[ A_{k, 0} = B_{n-1} a^{n-1} - (A_0 B_{n-1} - A_1 B_{n-1}) a^{n-2} z + \ldots + (A_0 B_{n-1} - A_1 B_{n-1}) a^{n-2} z^3 + \ldots \]

The successive formation of the above components for \( k = 2, 3, 4, \ldots \) will call linear algorithms, for the degree of approximation is proportional to the number \( k \).

2. Specializing for \( k = 2 \), we obtain as components of second order;

\[
\begin{align*}
1 \cdot a^{n-2} + (n - 1) a^{n-2} z; \\
2 \cdot a^{n-8} + (n - 2) a^{n-8} z; \\
\vdots \\
\h a^{n-3} + (n - h) a^{n-3} z; \\
\vdots \\
(n - 2) a^{n+1} + 2 a^z; \\
(n - 1) a^n + 1 z; \\
na^{n-1}.
\end{align*}
\]

The ratio of the last two components is the well known method of Newton:

\[ a_1 = \frac{(n - 1) a^n + z}{na^{n-1}}, \]

which was reproduced by Mr. Evans in the Analyst of January, 1876.—

Of all the \( n \) different values for \( \sqrt[\_]{z} \), furnished by the components of second order, one will be the best, independent of \( z \) and \( a \), and this is the one where \( h = \frac{1}{2}(n - 1) \); i.e.

\[ a_1 = a \cdot \frac{(n - 1) a^n + z(n + 1)}{(n + 1) a^n + z(n - 1)} \]

where \( a \) denotes any convenient initial value, and \( a_1 \) the corrected value for \( \sqrt[\_]{z} \).—The method under (4) is of third order, and reappears among the \( n \) methods for \( k = 3 \).
3. Specializing for \( k = 3 \), we obtain the components of third order:

\[
\begin{align*}
1. a^{s-3} & \quad n^3 + 3n - 2.2 \quad a^{s-3} z + \frac{n-1}{2}(n-2) \quad a^{s-3} z^2; \\
3 \quad a^{s-3} & \quad n^3 + 5n - 6.2 \quad a^{s-4} z + \frac{n-2}{2}(n-3) \quad a^{s-4} z^2; \\
1.2 \quad a^{s-3} & \quad n^3 + 7n - 12.2 \quad a^{s-5} z + \frac{n-3}{2}(n-4) \quad a^{s-5} z^2; \\
\vdots & \quad \vdots \\
\frac{n(n+1)}{2} \quad a^{s-(s+2)} & \quad n^3 + 2n + 1 \quad n - n(n+1) \quad 2 \quad a^{s-(s+2)} z \\
\vdots & \quad \vdots \\
\frac{n-1}{2} \quad a^{s-(-s+1)} & \quad (n+1) \quad a^{s-(-s+1)} z; \\
\frac{n+1}{2} \quad a^{s-(-s+2)} & \quad (n-1) \quad a^{s-(-s+2)} z. \\
\end{align*}
\]

The last two components furnish the method (4) again after a slight reduction. The method under (4) seems to be the most practical of all, considering its simple form and rapidity of approximation. If the initial value \((a)\) has any number of correct decimals, the next corrected value has three times this number of correct decimals.

4. By fixing any value of \( k \) and repeating with any of the \( n \) possible methods the same process with the number \( k \), we have \( n \) different algorithms of the order \( k \). For \( k = 2 \) we double with every step the number of correct decimals; for \( k = 3 \) we multiply the number of correct decimals by 3, and so on.---Our general principle furnishes methods of any required degree of approximation.

If we would avoid raising to high powers, we have to prepare the given number \( z \) by proper multiplication so that its value is nearly unity. In this case 1 is a good initial value. Then form all the components of the second order and its \( n \) algorithms, and take the arithmetical mean of them. This value will multiply the number of correct decimals of the initial value by four.

A theorem still more general than the one here explained, and numerical examples, I am obliged to suppress here for want of room.

[Dr. Eggers writes under date of May 18th, "The case of a revolving ellipsoid of three unequal axes is treated of in Kirchhoft's Vorlesungen uber Mathematische Physik, (Leipzig, editor Teubner, 1876.) Vorlesung 25; and by Dirichlet in Abhandlungen der Koniglichen gesellschaft der Wissenschaften zu Gottingen, volume 8, 1860; and Rankine treats of it in London Philos. Transactions 1863, Part I, p. 227-"]
THEORY OF PARALLELS.

BY PROF. W. W. JOHNSON, ANNAPOLIS, MARYLAND.

It is well known that Euclid was obliged to make an assumption in establishing the doctrine of parallel lines. This assumption he made in the form of an axiom, and the attempts of his commentators to justify this assumption have been shown covertly to involve a still greater one, namely, in some form or other, the doctrine of direction of which that of parallels is a part. This has been made the excuse for adopting as a definition "lines which have the same direction" the use of which phrase, according to a strict construction, assumes that lines which make equal angles with (that is differ equally in direction from) one secant line make equal angles with any other secant line.

In 1834 a demonstration of Euclid's axiom appeared anonymously in Crelle's Journal, and it is my object in this note to reproduce this demonstration which seems to have been generally overlooked by writers of geometrical text-books, though apparently exactly what is needed to put the theory upon a perfectly sound basis. But before proceeding to the demonstration a few words on the exact position of Euclid's axiom in the theory.

It being impossible to establish directly that lines in a common plane which make equal angles with one secant line likewise make equal angles with any other secant (of which it would be an easy corollary that such lines cannot meet), the idea of non-meeting lines is introduced and it is to be proved that this notion is co-extensive with that of making equal angles with a secant line. Consider now the four connected propositions whose logical form is

I. Every case of \( A \) is a case of \( B \).
II. \( " " " " \text{not } A " " " " \text{not } B \).
III. \( " " " B " " " A \).
IV. \( " " " \text{not } B " " " \text{not } A \);

in which \( A \) is the notion of making equal angles with a secant line, and \( B \) is the notion of non-meeting how far so ever produced. I and IV follow logically one from the other; so likewise do II and III. In this case there is no difficulty in proving I and IV, the difficulty being in demonstrating II the "opposite" of I; namely that lines making unequal angles with a given line must meet. Now the "opposite" II (and hence the converse III) follows from I whenever the notion \( B \) is unique; that is if a case of \( A \) exists and has been proved to be a case of \( B \), there being but one case of \( B \); then the case of \( B \) is a case of \( A \), for otherwise the case of \( A \) would
furnish a second case of $B$, which is impossible. For instance, when it has been proved that the bisector of an angle of a triangle cuts the base in the ratio of the adjacent sides, the "opposite" and converse follow at once, because there is but one point in which a line can be cut in a given ratio. Now Euclid's assumption was the uniqueness of the notion $B$; that is, "through a given point but one line can be drawn parallel to (i.e., so as not to meet) a given line." The demonstration in Crelle of this proposition is a direct demonstration of II above, I being supposed previously established; it is substantially as follows:

Let $AB$ and $CD$ be line making equal angles with the secant $AE$, so that $AB$ and $CL$ cannot meet, and let $AF$ be any other line in the plane passing through $A$; then $AF$ and $CD$ will meet on that side of $AE$ on which the interior angles are less than two right angles.

From $A$ draw a series of lines cutting off from $FAE$ successive angles equal $BAF$, until we come to a line $AK$ falling on the other side of $AE$, and let $BAK$ contain $n$ angles equal to $BAF$. Then, as the successive angles may be shown equal by coincidence, the space $BAK$ (the lines being produced indefinitely) is $n$ times the space $BAF$. Also draw a series of lines making the same angles with $AE$ that $AB$ makes, and cutting off from $AE$ successive distances equal to $AC$, and let $AM$ contain $n$ distances equal to $AC$. Then the spaces $BACD$, $DCHL$, &c. may be shown equal by coincidence, therefore the space $BAMN$ equals $n$ times the space $BA-CD$. Now the space $BAK$ is greater than the space $BAMN$ therefore the space $BAF$ is greater than $BACD$. Now if $AF$ did not meet $CD$ the space $BAF$ would be less than the space $BACD$. Hence $AF$ will if sufficiently produced meet $CD$.

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PERFECT CUBES.

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BY PROF. D. M. SENSENIG, INDIANA, PA.

The object of this article is to resolve the general cubic polynomial $ax^3 + bx^2 + cx + d$, into special forms that may be rendered perfect cubes for particular values of $x$. 
Put \( ax^3 + bx^2 + cx + d = (px + q)^3 \),

then \((a - p^4)x^3 + (b - 3p^3q)x^2 + (c - 3pq)x + d - q^3 = 0 \).

**Sup. 1st.** Put \( q^3 = d = d_1^3 \), or \( q = d_1 \),
then \((a - p^4)x^3 + (bx + d_1^3)(px + d_1^3) = \cdots \) (A)
and \((a - p^4)x^3 + (b - 3p^3d_1)x + c - 3pd_1^3 = 0 \). \( \cdots \) (X)

**Sup. 2nd.** Put \( p^3 = a = a_1^3 \), or \( p = a_1 \),
then \((a_1^3 + bx^2 + cx + d = (a_1x + q^3) \), \( \cdots \) (B)
and \((b - 3a_1^3q)x^3 + (c - 3a_1q)x + d - q^3 = 0 \). \( \cdots \) (Y)

**Sup. 3rd.** Put \( 3p^3q = b, \) or \( 3p^3q^3 = 3b_1^3, \) or \( p = b_1 + q_1 \),
then \((a^3 + bx^3 + cx + d = (b_1x + q^3) \), \( \cdots \) (C)
and \((a - b_1^3)x^3 + (c - 3b_1q^3)x + d - q^3 = 0 \). \( \cdots \) (Z)

**Sup. 4th.** Put \( 3pq^3 = c, \) or \( 3p^3q^3 = 3c_1^3, \) or \( q = c_1 + p_1 \),
then \((a^3 + bx^2 + 3c_1^3x + d = (p_1 + q^3) \), \( \cdots \) (D)
and \((a - p_1^3)x^3 + (b - 3p_1q)x^2 + d - c_1^3 + p_1^3 = 0 \). \( \cdots \) (W)

I. Resuming equations \( X \) and \( A \):

1st. Put \( 3pd_1^3 = c, \) or \( p = c + 3d_1^3, \) in (X) and reduce, then
\[
x = \frac{9c_2d_1^9 - 27bd_1^6}{27ad_1^6 - c^3}, \text{ and } ax^3 + bx^2 + cx + d_1^3 = \left(\frac{-2c_2d_1^9 - 9bd_1^6c + 27ad_1^6}{27ad_1^6 - c^3}\right)^3.
\]
\[
\therefore x = \frac{9c_2d_1^9 - 27bd_1^6}{27ad_1^6 - c^3}\text{ renders } ax^3 + bx^2 + cx + d = \left(\frac{2c_2d_1^9 - 9bd_1^6c + 27ad_1^6}{27ad_1^6 - c^3}\right)^3. \quad (A')
\]
Cor. Put \( 2cd - 9bd'c + 27ad' = 0 \), or \( b = (2c + 27ad') / 9ad' \), in (A') and reduce, then
\[
x = \frac{3c_2d_1^9 - 81ad_1^6}{279ad_1^6 - c^3}\text{ renders } ax^3 + \frac{2c_2 + 27ad'}{9ad'}x^2 + cx + d_1 = 0. \quad (a')
\]

2nd. Put \( p^3 = a = a_1^3, \) or \( p = a_1 \) in (X), and reduce, then
\[
x = \frac{3a_1d_1^3 - c}{b - 3a_1^2d_1}, \text{ and } a_1^3x^3 + bx^2 + cx + d_1^3 = \left(\frac{bd_1 - c}{b - 3a_1^2d_1}\right)^3.
\]
\[
\therefore x = \frac{3ad_1^3 - c}{b - 3ad_1^3}\text{ renders } ax^3 + bx^2 + cx + d = \left(\frac{bd_1 - c}{b - 3ad_1^3}\right)^3. \quad (A'')
\]
Cor. Put \( bd - c = 0, \) or \( c = bd, \) in (A''), and reduce, then
\[
x = \frac{3ad_1^3 - bd}{b - 3ad_1^3}\text{ renders } ax^3 + bx^2 + bdx + d_1 = 0. \quad (a'')
\]
3rd. Put \( 3p^3 d_1 = b \), or \( 3p^3 = 3b_1^3 + d_1^3 \), or \( p = b_1 + d_1 \), in \((X)\) and reduce, then
\[
x^3 = \frac{3b_1 d_1^2 - cd_1}{ad_1^2 - b_1^3} = \frac{d_1^2}{(ad_1^2 - b_1^3)^2} \left[ (3b_1 d_1^2 - cd_1)(ad_1^2 - b_1^3) \right];
\]
whence \( x = d_1 \sqrt[3]{(3b_1 d_1^2 - cd_1)(ad_1^2 - b_1^3)} + (ad_1^2 - b_1^3) \).

Put \( 3b_1 d_1^2 - cd_1 = n^2(ad_1^2 - b_1^3) \), or \( c = [3b_1 d_1^2 - n^2(ad_1^2 - b_1^3)] + d_2 \),
then \( x = nd_2 \), and \( ax^3 + 3b_1^2 x^2 + [3b_1 d_1^2 - n^2(ad_1^2 - b_1^3)] x + d_2 + d_1^2 \)
\(= (b_1 n + d_2)^3 \).

\( \therefore x = nd \) renders \( ax^3 + 3b^2 x^2 + [3bd^2 - n^3(ad^2 - b^3)] x + d + d^2 \)
\(= (bn + d)^3 \).

Cor. Put \( bn + d^2 = 0 \), or \( n = -d^2 \div b \), in \((A')\) and reduce, then
\( x = -d^2 \div b \) renders \( ax^3 + 3b^2 x^2 + (4b^3 d^2 - ad^2)x + b^2 d + d^3 = 0 \).

II. Resuming equations \((Y)\) and \((B)\):

1st. Put \( q^2 = d = d_1^2 \), or \( q = d_1 \), in \((Y)\) and reduce, then
\[
x = \frac{3a_1 d_1^2 - c}{b - 3a_1^2 d_1}, \text{ and } a_1^2 x^3 + bx^2 + cx + d_1^2 = \left( \frac{bd_1 - a_1 c}{b - 3a_1^2 d_1} \right)^3.
\]

\( \therefore x = \frac{3ad^2 - c}{b - 3a^2 d} \) renders \( a^2 x^3 + bx^2 + cx + d^3 = \left( \frac{bd - a c}{b - 3a^2 d} \right)^3 \).\((B')\)

Cor. Put \( bd = ac \), or \( c = bd \div a \), in \((B')\) and reduce, then
\[
x = \frac{3a^2 d^3 - bd}{ab - 3a^2 d} \text{ renders } a^2 x^3 + bx^2 + \frac{bd}{a} x + d^3 = 0.
\]

2nd. Put \( 3a_1 q = b \), or \( q = b \div 3a_1 \), in \((Y)\) and reduce, then
\[
x = \frac{b^3 - 27a_1^2 d}{27a_1^2 c - 9a_1^2 b}, \text{ and } a_1^2 x^3 + bx^2 + cx + d = \left( \frac{9a_1^2 bc - 2b^3 - 27a_1^3 d}{27a_1^2 c - 9a_1^2 b} \right)^3.
\]

\( \therefore x = \frac{b^3 - 27a^2 d}{27a^2 c - 9a^2 b^2} \) renders \( a^2 x^3 + bx^2 + cx + d = \left( \frac{9a^2 bc - 2b^3 - 27a^3 d}{27a^2 c - 9a^2 b^2} \right)^3 \).

Cor. Put \( 9a^2 bc - 2b^3 - 27a^3 d = 0 \), or \( d = (9a^2 bc - 2b^3) \div 27a^3 \), in \((B'')\) and reduce, then
\[
x = \frac{b^3 - a_2^3 bc}{9a^2 c - 3a^2 b^3} \text{ renders } a^2 x^3 + bx^2 + cx + \frac{9a^2 bc - 2b^3}{27a^3} = 0.
\]

3rd. Put \( 3a_1 q^2 = c \), or \( 3a_2^2 q^2 = c_1 \), or \( q = c_1 - a_2 \), in \((Y)\) and reduce, then
\[
x^3 = \frac{c_1^3 - a_2^3 d}{a_2^2 b - 3a^2 c_1} = \frac{1}{a_2(a_2 b - 3a^2 c_1)} \left[ (c_1^3 - a_2^3 d)(a_2 b - 3a^2 c_1) \right];
\]
whence \( x = \frac{1}{a_2(a_2 b - 3a^2 c_1)} \sqrt{\left( c_1^3 - a_2^3 d \right)(a_2 b - 3a^2 c_1)} \).
Put \( c_1^2 - a_2^2d = n^2(a_2b - 3a_4c_1) \), or \( d = [c_1^2 - n^2(a_2b - 3a_4c_1)] \div a_2^2 \),
then \( x = \frac{n}{a} \) renders \( a_2x^2 + bx^3 + 3c^2x + \frac{c_1^2 - n^2(a_2b - 3a_4c_1)}{a_4^2} = \left(\frac{a_2^2n + c_1}{a}\right)^3 \).

\[ \therefore x = \frac{n}{a} \] renders \( a_2x^3 + bx^2 + 3c^2x + \frac{c_1^2 - n^2(ab - 3a_4c)}{a^3} = \left(\frac{a^2n + c}{a}\right)^3. \]

Cor. Put \( a^2n + c = 0 \), or \( n = -\frac{c}{a^2} \), in \((B')\) and reduce, then
\[ x = -\frac{c}{a^3} \] renders \( a_2x^3 + bx^2 + 3c^2 + \frac{c^1 - abc + 3a_4c^2}{a^7} = 0 \ldots (b''') \)

III. Resuming equations \((Z)\) and \((C)\):

1st. Put \( b_1^2 = aq_1^2 = a_2^2p_1^2 \), or \( q_1 = b_1 + a_1 \), in \((Z)\) and reduce, then
\[ z = \frac{b_1^2 - a_2^2d}{a_1c - 3a_2b_1^2} \] and \( a_2x^2 + 3b_1^2x^3 + cx + d = \left(\frac{a_2b_1c - 2b_1 - a_4d}{a_1c - 3a_2b_1^2}\right)^3 \).

\[ \therefore x = \frac{b_1^2 - a_2^2d}{a_1c - 3a_2b_1^2} \] renders \( a_2x^3 + 3b_1^2x^2 + cx + d = \left(\frac{a_2bc - 2b_1 - a_4d}{a_1c - 3a_2b_1^2}\right)^3 \).

Cor. Put \( a_2bc - 2b_1 - a_4d = 0 \), or \( d = (a_2bc - 2b_1) + a_4 \), in \((C')\) and reduce, then
\[ x = \frac{3b_1 - a_2^2b_1}{a_1c - 3a_2b_1^2} \] renders \( a_2x^3 + 3b_1x^2 + cx + \frac{a_2bc - 2b_1}{a_4} = 0 \ldots (c') \)

2nd. Putting \( q_1 = d = d_1^2 \), or \( q_1 = d_1 \), in \((Z)\) and \((C)\) produces \((A'')\) and hence will be omitted.

3rd. Put \( 3b_1q_1^2 = c \), or \( 3b_1^2q_1^2 = 3c_1^2 \), or \( q_1 = c_1 + b_2 \), in \((Z)\) and reduce, then
\[ x^3 = \frac{c_1^2 - b_2^2c_1d}{ab_2c^2 - b_2^3} = \frac{c_1^2}{b_2^2(c_1^2 - b_2^2d)} \left[ (c_1^2 - b_2^2d)(a_2c_1^2 - b_2^2) \right]^{1/3} \];
whence \[ x = \frac{c_1^2}{b_2^2(c_1^2 - b_2^2d)} \left[ (c_1^2 - b_2^2d)(a_2c_1^2 - b_2^2) \right]^{1/3} \].

Put \( c_1^2 - b_2^2d = n^2(a_2c_1^2 - b_2^2) \), or \( d = [c_1^2 - n^2(a_2c_1^2 - b_2^2)] + b_2^2 \); then
\[ x = \frac{c_1n}{b_2^2} \] and \( ax^3 + 3b_2^2x^2 + 3c_2^2x + \frac{c_1^2 - n^2(a_2c_1^2 - b_2^2)}{b_2^2} = \left(\frac{b_2n + c_2}{b_2}\right)^3 \).

\[ \therefore x = \frac{cn}{b_2} \] renders \( ax^3 + 3b_2^2x^2 + 3c_2^2x + \frac{c_1^2 - n^2(a_2c_1^2 - b_2^2)}{b_2^2} = \left(\frac{b_2n + c_2}{b_2}\right)^3 \).

Cor. Put \( b_2n + c_2 = 0 \), or \( n = -\frac{c_2}{b_2} \), in \((C')\) and reduce, then
\[ x = -\frac{c_2}{b_2} \] renders \( ax^3 + 3b_2^2x^2 + 3c_2^2x + \frac{c_1^2 - n^2(a_2c_1^2 - b_2^2)}{b_2^2} = 0 \ldots (c') \)

IV. Resuming equations \((W)\) and \((D)\):

1st. Put \( d = c_1^2 + p_2^2 \), or \( p_2^2d_1^2 = c_1^2 \), or \( p_1 = c_1 + d_1 \), in \((W)\) and reduce, then
\[ x = \frac{3c_1d_1^2 - bd_1}{ad_1^2 - c_1^2} \quad \text{and} \quad ax^3 + bx^2 + 3c_1x + d_1 = \left(\frac{2c_1d_1 - bd_1}{ad_1^2 - c_1^2}\right)^2. \]

.. \[ x = \frac{3c_1^2d_1^3 - bd_1^3}{ad_1^3 - c_1^3} \quad \text{renders} \quad ax^3 + bx^2 + 3cx + d = \left(\frac{2c_1d_1 - bd_1}{ad_1^2 - c_1^2}\right)^2. \]

... (D')

Cor. Put \(2c_1d - bc'd + ad' = 0\), or \(b = (2c_1d + ad') + cd', \) in (D') and reduce, then

\[ x = \frac{c_1d^2 - ad^2}{ad^2 - d'} \quad \text{renders} \quad ax^3 + \frac{2c_1d + ad'}{cd} \quad x^3 + 3cx^3 + d^3 = 0. \ldots (d') \]

2nd. Put \(p_1 = a = a_4\), or \(p_1 = a_1\), in (\(W\)) and reduce, then

\[ x = \frac{c_1^3 - a_1^2d}{a_1^2b - 3a_1^2c_1} = \frac{1}{(a_1^2b - 3a_1^2c_1)} \left( (c_1^2 - a_1^2d)(a_1^2b - 3a_1^2c_1) \right); \]

whence \[ x = \frac{1}{a_1^2b - 3a_1^2c_1} \left( (c_1^2 - a_1^2d)(a_1^2b - 3a_1^2c_1) \right). \]

Put \(c_1^2 - a_1^2d = n^2(a_1^2b - 3a_1^2c_1)\), or \(d = [c_1^2 - n^2(a_1^2b - 3a_1^2c_1) + a_1^2] \), then \(x = n\), and \(a_1^2x^3 + bx^2 + 3c_1^2x + \frac{c_1^2 - n^2(a_1^2b - 3a_1^2c_1)}{a_1} = \left(\frac{a_1^2n + c_1}{a_1}\right)^2. \)

.. \[ x = n \quad \text{renders} \quad ax^3 + bx^2 + 3cx^2 + \frac{c_1^2 - n^2(a_1^2b - 3a_1^2c_1)}{a_1} = \left(\frac{a_1^2n + c_1}{a_1}\right)^2. \]

... (D'')

Cor. Put \(a^2n + c = 0\), or \(n = -c + a^2\), in (D'') and reduce, then

\[ x = -c + a^2 \quad \text{renders} \quad ax^3 + bx^2 + 3cx^2 + (4a^2n^2 - bc') + a^2 = 0. \]

... (d'')

3rd. Put \(3p_1^2c_1 = b = 3b_1\), or \(p_1 = b_1 + c_3\), or \(p_1 = b_1 + c_3\), in (\(W\)) and reduce, then

\[ x = \frac{c_1^2}{b_1(4c_2^2 - b_1^2)} = \frac{c_1^2}{b_1(4c_2^2 - b_1^2)} \left( \frac{c_1^2 - b_1^2d}{ac_2^2 - b_1^2} \right)^2; \]

whence \[ x = \frac{c_1^2}{b_1(4c_2^2 - b_1^2)} \left( \frac{c_1^2 - b_1^2d}{ac_2^2 - b_1^2} \right)^2. \]

Put \(c_1^2 - b_1^2d = n^2(ac_2^2 - b_1^2)\), or \(d = [c_1^2 - n^2(ac_2^2 - b_1^2) + b_1^2] \), then

\[ x = \frac{c_1^2n}{b_1}, \quad \text{and} \quad ax^3 + 3b_1^2x^2 + 3c_2^2x + \frac{c_1^2 - n^2(ac_2^2 - b_1^2)}{b_1} = \left(\frac{b_1^2n + c_1^2}{b_1}\right)^2. \]

.. \[ x = \frac{c_1^2n}{b}, \quad \text{renders} \quad ax^3 + 3b_1^2x^2 + 3c_2^2x + \frac{c_1^2 - n^2(ac_2^2 - b_1^2)}{b_1} = \left(\frac{b_1^2n + c_1^2}{b_1}\right)^2. \]

... (D''')

Cor. Put \(b'n + c' = 0\), or \(n = -c' + b'\), in (D''') and reduce, then

\[ x = -c' + b' \quad \text{renders} \quad ax^3 + 3b_1^2x^2 + 3c_2^2x + ac_1^2 + b' = 0. \ldots (d''') \]
LAND SURVEYING.

BY THE EDITOR.

The object of this article is to point out some of the practical difficulties encountered in ordinary land surveying.

The instruments used, in this country, in land surveying are, the Surveyor's Compass, the Surveyor's Transit, and the Solar Compass; the merit of each of which depends upon the kind of work to be done and the locality of the survey. When it is required to do the work in the least practicable time and with but ordinary accuracy, and especially in a timbered country, the surveyor's compass, on a tripod with ball and socket mounting, is perhaps the best. But in open country, and if a considerable degree of accuracy is required, the surveyor's transit is by far the best.

The Solar Compass is chiefly used, in this country, for surveying base lines, meridians, and the exterior township lines in the government surveys, for which purpose the Commissioner of the general land office, in his instructions to Surveyors General of public lands, directs it to be used.

The chief practical difficulty in making surveys with a good instrument of either of the first two kinds mentioned above is the local, and the daily variation of the magnetic needle. The local variation may be temporary or permanent. Against the first of these the vigilance of the surveyor may be sufficient to guard, and of the second he should be notified by the record of the original survey as made with the solar compass. But as such record is not always to be found, and not always reliable when found, the surveyor should determine the variation at as many points as practicable by astronomical observation, and supplement his knowledge of any local variation in the vicinity where his work may chiefly lie by transit lines between the points of observation.

The daily variation of the needle, although its range is less extended, is, in general, more embarrassing to the surveyor than the local variation, for, although it seldom exceeds a fourth of a degree, yet as it is irregular he has no means of knowing its exact value at any time unless he can compare his bearing with some known course, which is not in general practicable in ordinary surveying.

Besides the variations of the magnetic needle above alluded to, and besides the secular variation, which is different for different places, there is an occasional variation from meteoric disturbances, or "magnetic storms," from which I have noticed a temporary deflection of the needle through more than a degree of arc.
Considering all these sources of error, to say nothing of the errors of observation and measurement, it will readily be seen that the results obtained in ordinary land surveying are but rough approximations towards accuracy.

Theoretically the Solar Compass obviates the errors above alluded to, and to which a reliance on the magnetic needle gives rise. Practically however there are inconveniences and difficulties attending the use of the solar compass, so that, in ordinary land surveying, and after permanent local variations have been determined, it is doubtful whether the same amount of work can be done, in a given time, with the solar compass with more accuracy than with the ordinary compass or transit.

The principal objection to the use of the solar compass in an open country results from the circumstance that it cannot be used with ordinary accuracy when the sun is within fifteen degrees of the meridian, nor again when the sun is near the horizon, which, in connection with the fact that the sun is frequently obscured by clouds when a sight by the compass is desired, so far curtails the time at the disposal of the surveyor that the average length of a day's work with the solar compass falls far short of a day's work with the ordinary compass or transit. And, in addition to the loss of time from the causes above enumerated, the instrument must be accurately leveled at every station, otherwise the course indicated will be in error; the use of a ball and socket mounting for the tripod of a solar compass is therefore entirely impracticable, and hence when short sights have to be taken, which is unavoidable in a timbered or hilly country, much time is occupied in leveling the instrument. To these sources of delay and error must be added the time required for the frequent adjustment of the Latitude and Declination arcs of the instrument, and the error resulting from the unequal refraction of the sun's rays by the atmosphere at different times and places.

Notwithstanding, however, all these objections and sources of error, the solar compass can be used with much advantage and considerable accuracy, by a skilful surveyor, in an open country and where long sights can be taken.

It is not my purpose in this paper to discuss the office work of the surveyor, which includes "supplying omissions" in the survey; this, however, has been ably treated in these pages by Professors Philbrick and Abbe. (See Analyst, Vol. II, pp. 116 and 182.) But as it is desirable that the field work of the surveyor should not be interrupted by trigonometrical calculations, I add the following description of an easy method of continuing the measurement of a line across a stream or pond that cannot be chained, without any calculation:
Let $P$ represent the stream or pond and $BA$, a portion of the line. With the instrument at the station $S$, set a flag at $A$, on the opposite side of the stream or pond, and also set a flag at $D$, any convenient point on the line, in the opposite direction from $S$. Take any point $C$, at right-angles with the line $AB$ from the station $S$ and at any convenient distance, which need not be measured. Set the instrument at the station $C$ and direct the sights to the flag at $A$ and note the angle $SCA$; turn the instrument towards a flag to be placed at a point $B$, which must be kept in range with the flag at $D$ by an assistant at $S$, making the angle $SCB$ equal the angle $SCA$.

Now because of the similar triangles $SCA$ and $SCB$ having a common side $CS$, we have $SB = SA$; therefore chain back from $S$ to $B$, and the result will be the length of the line produced to $A$.

When a back sight to $B$ is not practicable the equivalent of the line $SA$ may be chained by constructing on $SA$ an equilateral triangle one side of which shall lie wholly on the opposite side of the stream.

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**QUERY. BY THE EDITOR.** If a cube be cut by four vertical planes each of which intersects two contiguous vertical faces along their center lines, thus cutting away half the volume of the cube; and if the two ends of the remaining parallelopipedon be each cut by four planes, each of the four planes passing through the center of the upper and the lower base, respectively, making an angle of $45^\circ$ with the vertical axis and intersecting two contiguous sides of the parallelopipedon at equal angles, thus cutting off the eight corners of the parallelopipedon, the remaining solid is a crystalline form known to mineralogists as a “rhombic dodecahedron,” (Dana,) having twelve faces and fourteen solid angles. Through the center of this solid let a plane be passed which shall intersect its six parallel edges at right-angles, and the solid will be divided into two equal and similar solids which are identical in form with the honey bee’s cell; that is, each will be a hexagon terminated by three planes which make respectively an angle of $54^\circ44'8''$ with the axis of the hexagon. Though this is the angle which gives a minimum surface for a given volume, and therefore would seem to indicate intelligence and design in its construction, yet being a crystalline form, may not the physical forces which determine the position of the molecules in the formation of the rhombic dodecahedron influence the action of the bee in the construction of its cell rather than the intelligence and design of the bee?
RECENT MATHEMATICAL PUBLICATIONS.

COMMUNICATED BY G. W. HILL.

Todhunter, I.—An Elementary Treatise on Laplace's Functions, Lamé's Functions and Bessel's Functions. New York, 1875. Macmillan. 8vo. 346 pp. $4.00

Whewell, Dr. William, Master of Trinity College, Cambridge.—An account of his writings, with selections from his literary and scientific correspondence. By I. Todhunter. New York, 1876. Macmillan. 2 vols. 8vo. $10.00


Mansion, Paul.—Théorie des équations aux dérivées partielles du premier ordre. Bruxelles, 1875. 8vo. 289 pp. 6 fr.


Hansen, P. A.—Über die Störungen der grossen Planeten, ins bes ondere des Jupiter. Leipzig, 1875. Hirzel. 8vo. 204 pp. 6 M.


Riemann, Bh.—Schwere, Elektricität und Magnetismus. Nach den Vorlesungen bearbeitet von K. Hattendorf. Hanover, 1876. Rümpler. 8vo. 358 pp. 8 M.


Hankel, H.—Die Elemente der projectivischen Geometrie in synthetischer Behandlung. Leipzig, 1876. 8vo. 256 pp. 7 M.
LIMITS OF THE PRISMOIDAL FORMULA.

BY PROF. E. W. HYDE, CINCINNATI, OHIO.

Consider the solid represented by the following integral:

\[ V = \int_{x=0}^{x=a} \int_{y=0}^{y=f(x)} \int_{z=0}^{z=\psi(x,y)} dx \, dy \, dz \quad \cdots \cdots \quad (1) \]

If the Prismatic formula holds for this volume, we shall ascertain the fact by comparing the result of the integration of (1) with the result obtained by placing in the formula

\[ V = \frac{\pi}{6} \left[ A_0 + 4A_{\alpha} + A_{\beta} \right] \]

(in which \( A_0 \) is the section by the plane \( YZ \), \( A_{\alpha} \) that by the plane \( x=\frac{1}{3}a \&c.) \) the values of the integral

\[ A = \int_{y=0}^{y=f(x)} \int_{z=0}^{z=\psi(x,y)} dy \, dz \quad \cdots \cdots \quad (2) \]

when \( x \) is made successively equal to 0, \( \frac{1}{3}a \) and \( a \).

It is evident in the first place that the functions \( f(x) \) and \( \psi(x,y) \) must be such that, on the integration of (1) with reference to \( x \), no new function shall be introduced different from what existed previously, such as a logarithm; for as this would not appear in (2), the results obtained by integration, and by the formula could not agree. It will be seen also that in (1) the result of the first two integrations must not contain radicals involving \( x \), since on integrating again these would be raised to higher powers, which would not be done by the formula.

Let us then assume \( \psi(x,y) \) a rational integral expression of the form

\[ z = \psi(x,y) = \frac{Ax^n y^n + Bx^m y^{m-1} + Cx^{m-1} y^m + \cdots}{+ Ex^m + Fx^{m-1} + Gx^{n-1} + \cdots} \quad \frac{Ky^n + Ly^{n-1} + My^{n-2} + \cdots}{+ P} \quad \cdots \cdots \quad (3) \]

in which \( m \) and \( n \) are positive integers.

Since each term of this polynomial is to be integrated by itself we will consider only the term of highest degree \( Ax^ny^n \). Thus

\[ V_1 = A \int_0^a \int_0^{f(x)} x^n y^n \, dx \, dy = A \int_0^a \left[ \frac{x^n y^{n+1}}{n+1} \right]_0^{f(x)} \, dx \quad \cdots \cdots \quad (4) \]

Let \( y = f(x) = A_1 x^p + B_1 x^{p-1} + \cdots + Q \quad \cdots \cdots \quad (5) \)

and as before let us consider only the first term \( A_1 x^p \); then

\[ V_2 = \frac{A A_1^{p+1}}{n+1} \int_0^a x^{(n+1)+mp} \, dx = \frac{A A_1^{p+1}}{n+1} \left( \frac{x^{(n+1)+m+1}}{p(n+1)+m+1} \right) \quad \cdots \cdots \quad (6) \]
Now let us apply the Prismoidal Formula. From equation (2)
\[ A_0 = \int_{y=0}^{y=y(0)} \int_{x=0}^{x=x(0,y)} dV dy dx = \int_0^a (Kx^p + Lx^{p-1} + \ldots + P) dx = KQy^p + LQy^{p-1} + \ldots + PQ. \ldots \ldots \ldots \ldots (7) \]

If however we consider only the term integrated in (6) we have
\[ A_0 \cdot 2 = 0, \quad \text{and similarly} \]
\[ A_{n+2} = \frac{AA_1^{n+1}}{n+1} \cdot \frac{\alpha^{p(n+1)+m}}{\alpha^{p(n+1)+m}} \]
\[ A_{n+2} = \frac{AA_1^{n+1}}{n+1} \cdot \frac{\alpha^{p(n+1)+m}}{\alpha^{p(n+1)+m}} \]

\[ \cdots \]
\[ V_2 = \frac{a}{6} \left[ \frac{4AA_1^{n+1}}{n+1} \cdot \frac{\alpha^{p(n+1)+m}}{\alpha^{p(n+1)+m}} + \frac{1}{n+1} \cdot \frac{\alpha^{p(n+1)+m}}{\alpha^{p(n+1)+m}} \right] \]
\[ = \frac{AA_1^{n+1}}{n+1} \left[ \frac{\alpha^{p(n+1)+m+1}}{3 \times 2^{p(n+1)+m-1}} + \frac{1}{6} \right] \ldots (8) \]

The difference between the results obtained by the formula and by integration is
\[ V_2 - V_2 = AA_1^{n+1} \cdot \frac{\alpha^{p(n+1)+m+1}}{n+1} \left[ \frac{1}{3 \times 2^{p(n+1)+m-1}} + \frac{1}{6} - \frac{1}{p(n+1)+m+1} \right]. \]

If the quantity in the parenthesis is zero the prismoidal formula must be correct. In the parenthesis let
\[ p(n+1) + m = 1, \quad \text{then } \frac{1}{6} + \frac{1}{6} = \frac{1}{3} = 0; \]
\[ p(n+1) + m = 2, \quad \text{" } \frac{1}{6} + \frac{1}{6} = \frac{1}{3} = 0; \]
\[ p(n+1) + m = 3, \quad \text{" } \frac{1}{6} + \frac{1}{6} = \frac{1}{3} = 0; \]
\[ p(n+1) + m = 4, \quad \text{" } \frac{1}{6} + \frac{1}{6} = \frac{1}{3} = 0; \]
\[ p(n+1) + m = 5, \quad \text{" } \frac{1}{6} + \frac{1}{6} = \frac{1}{3} = 0; \]
\[ p(n+1) + m = 6, \quad \text{" } \frac{1}{6} + \frac{1}{6} = \frac{1}{3} = 0. \]

Thus it appears that the formula is perfectly correct so long as \( p(n+1) + m + 1 \)

+ m does not exceed three, and that from this point on there is a constantly increasing error.

Let us consider the cases that arise when different values are assigned to p.

1st \[ p = 3, \quad \therefore m = n = 0; \]
2nd \[ p = 2, \quad \therefore m = 1, n = 0; \]
3rd \[ p = 1, \quad \therefore m = 0, n = 2, or m = 1, n = 1, or m = 2, n = 0; \]
4th \[ p = 0, \quad \therefore m = 3, n = any finite number. \]

In the first Case \( f(x) = A_1 x^3 + B_1 x^2 + C_1 x + Q = y, \)
and \( \varphi(x,y) = P = z, \) thus the upper surface is a horizontal plane.

In the 2nd Case \( f(x) = A_1 x^3 + B_1 x + Q = y, \)
and \( \varphi(x,y) = A' x + P = z, \) a plane perpendicular to ZX.
In the third case \( f(x) = A_1 x + Q = y \), and 
\( \phi(x, y) = A'x^4 + B'xy + C'x^2y + D'x^2y + E'y + P = z \),
an elliptic or hyperbolic paraboloid.

In the fourth case \( f(x) = Q = y \), and \( \phi(x, y) \) is a rational integral function of \( x \) and \( y \) in which \( y \) may appear with any exponent, but \( x \) must not be of higher degree than the 3rd. The surface represented by \( \phi(x, y) \) may then be made, in this case, to pass through any number of points, provided that not more than four are in any single plane parallel to \( ZX \).

Let us now determine the limits of Weddle's Formula for seven equidistant cross-sections.

\[
V = \frac{\alpha}{20} \left[ A_0 + 5A_{\alpha} + A_{2\alpha} + 6A_{3\alpha} + A_{4\alpha} + 5A_{5\alpha} + A_\alpha \right]. \tag{13}
\]

Proceeding precisely as before we find

\[
A_0 = 0,
\]

\[
5A_\alpha = \frac{5AA_1^{n+1} \alpha^{p(n+1)+m}}{n+1} \cdot \frac{6^{p(n+1)+m}}{6^{p(n+1)+m}}
\]

\[
A_2 = \frac{AA_1^{n+1}}{n+1} \cdot \frac{6^{p(n+1)+m}}{6^{p(n+1)+m}} \qquad \& \text{c.}
\]

\[
V_2' = \frac{AA_1^{n+1} \alpha^{p(n+1)+m+1}}{n+1} \times \left[ \frac{5 + 2^{p(n+1)+m} + 6 \times 3^{p(n+1)+m} + 4^{p(n+1)+m} + 5^{p(n+1)+m} + 6^{p(n+1)+m}}{20 \times 6^{p(n+1)+m}} \right]. \tag{14}
\]

Subtract (6) from (14) and let the difference be \( \frac{\delta AA_1^{n+1} \alpha^{p(n+1)+m+1}}{n+1} \).

Make \( p(n+1) + m \) equal successively to 1, 2, 3, &c.

\[
\delta_1 = 0, \quad \delta_2 = 0, \quad \delta_3 = 0, \quad \delta_4 = 0, \quad \delta_5 = 0,
\]

\[
\delta_6 = \frac{1}{54432}, \quad \delta_7 = \frac{1}{15552}, \quad \delta_8 = \frac{113}{699840} \quad \left( = \frac{1}{6193} \text{ nearly.} \right)
\]

Thus the formula is exact when \( p(n+1) + m \) does not exceed 5, and the error is small when \( p(n+1) + m \) is 6, or even 7.

The formula gives each term which is of too high degree a numerical value greater than that obtained by integration, but whether the final result is to be greater or less depends upon the signs of the coefficients of the terms of too high degree.

Giving different values to \( p \), we have, when \( p = 0 \), \( n = \) any number whatever, and \( m = 5 \) at most if the result is to be exact.

If \( p = 1 \), we have \( f(x) = A_1 x + Q \),
and 
\[
z = \phi(x, y) = Ax^4 + Bxy + Cx^2y^2 + Dxy^3 + Egy^4 + Fx^2 + Gx^2y + Hxy^3 + Iy^4 + Kx^2 + Lxy + My^3 + Nx + Oy + P.
\]
If \( p = 2 \),
\[
f(x) = A_1 x^2 + B_1 x + Q,
\]
and
\[
z = \phi(x, y) = Ax^2 + Bx + Cxy + Dy + Ey + P.
\]
If \( p = 3 \),
\[
f(x) = A_1 x^3 + B_1 x^2 + C_1 x + Q,
\]
and
\[
z = \phi(x, y) = Ax^2 + Bx + P,
\]
a parabolic cylinder.

If \( p = 4 \) the upper limiting surface becomes a plane parallel to \( Y \), and if \( p = 5 \), it is a plane parallel to \( XY \).

Let us now consider the wedge-shaped solid \( OA - BCDE \), using \( x, \theta \) as coordinates instead of \( x, y, z \), as shown in the figure. The volume is then
\[
V = \int_0^a \int_{\theta_0}^{\theta_1} \int_0^{\rho=\psi(x, \theta)} dx \rho d\theta d\rho = \frac{1}{2} \int_0^a \int_{\theta_0}^{\theta_1} \left[ \rho^2 \right]_0^{\psi(x, \theta)} dx d\theta \ldots (15)
\]
Let \( \rho = \psi_1(x, \theta) = \sqrt{[x^2 f_1'(\theta) + \ldots + 1]} \)
\[
f_1'(\theta) = \frac{d}{d\theta} \left[ f_1(\theta) \right].
\]

\( \ldots \) considering as before only the first term we have
\[
V_1 = \frac{1}{2} \int_0^a \int_{\theta_0}^{\theta_1} x^2 f_1'(\theta) dx d\theta = \frac{1}{2} \left[ f_1(\theta_2) - f_1(\theta_1) \right] \frac{a^{m+1}}{m+1} \ldots (17)
\]
Now both the Prismatic formula and that of Weddle hold for this case to the same extent as for that treated before, since we have only to make \( p \) equal 0 in (6) to reduce it to a form similar to (16).

Hence the prismatic formula holds when \( x \) enters no term of \( \phi(x, y) \) or \( \phi_1(x, \theta) \) to a degree higher than the third, and Weddle's when \( x \) enters no term to a degree higher than the fifth; while \( y \) or \( \theta \) may enter in any form algebraic or transcendental, so long as the second integration is between constant limits.

It is evident that these formulas hold to the same extent for the areas of plane curves in \( ZX \).

Weddle's formula seems decidedly preferable to the Prismatic for accuracy either for curves or solids.
NOTE ON THE POLYCONIC PROJECTION.

BY PROF. J. E. HILGARD, ASSISTANT IN CHARGE, U. S. COAST SURVEY.

In an article which appeared in the last January number of the Analyst (p. 15) Professor W. W. Johnson points out an error of statement respecting the description of the Polyconic Projection contained in the Coast Survey Report for 1853.

In referring to this subject I merely wish to remark that it was thought a sufficient corrective to publish, together with a concise statement of the principles and formulae of the polyconic projection, and extended tables, a map showing the whole globe on this projection on which the deviations from a right angle of the intersections of meridians and parallels are brought directly before the eye. This was done in the Coast Survey Report for 1856, plate No. 65, a reduced copy of which, herewith sent, may be interesting to your readers. In the subsequent Report for 1859, the diagram of the whole sphere was again given with a representation of the isogonic lines.

These illustrations appear to have escaped the notice of Prof. Johnson.
ON THE EXPRESSION $0^0$.

BY PROF. W. W. JOHNSON, ANNAPOLIS, MARYLAND.

It is now a well-established doctrine that $0^0$ is one of the indeterminate forms, since, when a function takes this form, its logarithm takes the form $0 \cdot \infty$: yet we almost invariably find that in such cases the value of the function is unity; and it is not surprising that, not many years ago, it was generally held that $0^0 = 1$ was a universally true equation. An interesting controversy on this point took place in Crelle's Journal about the year 1830.

Libri, in an Article on discontinuous functions (Crelle, Vol. X, p. 303), employs the expression $0^0$ as equivalent to unity. In support of this he asserts that $x \log 0$ (the log of $0^x$) is always 0 when $x = 0$, although the second factor is infinite. He admits that some uncertainty exists, since, admitting that $x \log x = 0$ when $x = 0$, "we do not know that the log 0 arises from $\log x$"; and he proceeds to remove this doubt by a direct proof that $0^0 = 1$. For this purpose he expands $(1 - u)^{0}$ by the Binomial Theorem, and remarking that for entire values of $x$ the development arrests itself after a finite number of terms by virtue of the occurrence of the factor $x - x$ in succeeding coefficients, he concludes that $(1 - u)^0 = 1$, independently of the value of $u$; and therefore when $u = 1$, we have $0^0 = 1$.

Libri's object in establishing $0^0 = 1$ is, by the way, a very curious one; he wishes to construct a function of $x$ which shall admit of but two values, one corresponding to all positive, and the other to all negative values of $x$. Taking $0^0 = 1$, he concludes that $0^x$ admits of but three values, 0, 1, and $\infty$, according as $x$ is positive, 0 or negative: hence $x^0$ can take only the values $0^0$, 0, 0, 0, or 1, 0, 0; that is, its value is 1 for all positive values of $x$, and 0 for $x = 0$ and all negative values of $x$. Libri had indeed previously published this result in Crelle, Vol. VI, p. 67.

In Vol. XI the author of Article 25, "Sur la valeur de $0^0$" (p. 272), takes exception to Libri's assumption that $0^0 = 1$, although he says that this is nearly the unanimous opinion of Mathematicians, Cauchy alone, among authors whom he has consulted, dissenting. He quotes from Euler the argument $\frac{a^0}{a} = a^0 = 1$ whatever the value of $a$, hence certainly when $a = 0$;" but the author says why not equally well say $a \div a = 1$, hence, making $a = 0$, $0 \div 0 = 1$ universally. [The argument is that, while we admit that, by the law of continuity, $a^0$ is unity when $a = 0$, or as we
may express it \( a^0 \) = 0, just as \( \frac{a}{a} \) = 1; we can no more infer from the former that 0\(^0\) = 1 universally, than from the latter that 0 ÷ 0 = 1 universally. The same remark applies to Libri's deduction from \((1 - x)^0 = 1\); reference to the Binomial Theorem adding nothing whatever to the strength of his position.] To prove 0\(^0\) = 1, he continues, it is necessary to show that, if \( F(x) \) and \( f(y) \) vanish for \( x = a \) and \( y = b \), \([F(x)]^{cx}\) becomes unity when \( x = a \) and \( y = b \). In his opinion 0\(^0\) is nothing more than 0\(^1\).0\(^{-1}\) or 0.\(\infty\) which may have any value.

In the next volume of Crelle published in 1834, Möbius gives (XII. p. 134) a proof, communicated to him in 1814 by Pfaff, that 0\(^0\) = 1, which proof he says in point of validity leaves nothing to be desired, and which he now publishes on account of the doubts, as to the universal truth of this equation, raised by the author of No. 25, Vol. XI. He begins by asserting that 0\(^0\) = 1 "means nothing but that \( x^x \) becomes unity when \( x = 0 \)." Then follows Pfaff's proof of this proposition; to which Möbius adds that, this being established, it is easy to show, as demanded by the author of No. 25, that, if \( F(x) \) and \( f(x) \) vanish simultaneously, \([F(x)]^{cx}\) will take the value unity: in other words, to show that, \( X \) and \( Y \) being functions of a third variable \( z \) which vanish when \( z = c \), or more simply when \( z = 0 \), \( X^Y \) becomes unity when \( z = 0 \).

To prove this he proceeds as follows:—Since for \( z = 0 \) \( X \) and \( Y \) vanish, they may be written in the forms,

\[
X = Pz^m, \quad Y = Qz^n,
\]

where \( m \) and \( n \) are positive and independent of \( z \), while \( P \) and \( Q \) are two functions of \( z \) which do not vanish with \( z \), but take certain finite values, say \( p \) and \( q \), when \( z = 0 \). Then

\[
X^Y = \left(\frac{P}{Q}\right)^z (z^m)^{Qz^n}.
\]

when \( z = 0 \) the first factor becomes \( \left(\frac{p}{q}\right)^0 = 1 \); and the second, by writing \( V \) for \( z^x \), becomes \( V^{\frac{m}{n}} QV = (V^V)^{\frac{m}{n}} \), which when \( z = 0 \), or \( V = 0 \), becomes (according to Pfaff's demonstration) \( 1^{\frac{m}{n}} = 1 \). Hence, when \( z = 0 \), the product or value of \( X^Y \) becomes unity.

In the same volume of Crelle, p. 292, we find two answers to Möbius, one "On einem Ungenaunten," the other by the author of No. 25. The latter remarks that no one denies that \( x^x \) becomes unity when \( x = 0 \), but if
we say that \(0^0 = 1\) means nothing more than this, we must also say that \(0^0 = 1\) means nothing more than that \(x^0 = x\) is unity when \(x = 0\). As to the demonstration of Möbius that \([F(x)]^{(n)}\) becomes unity when \(x = 0\), he replies that it rests upon a principle exploded by Hamilton, in the Transactions of the Royal Irish Academy, in 1830. The principle in question is that if \(X\) is a function which vanishes with \(x\) it may always be assumed in the form \(Px^n\), where \(P\) does not vanish with \(x\).

The Article cited from Hamilton is to the effect that it was a commonly received doctrine that a function of \(x\) which vanished with \(x\) could always be put in the form

\[ Ax^a + Bx^\beta + Cx^\gamma \ldots . \]

in which \(a, \beta, \gamma, \&c.\) are positive and \(A, B, C, \&c.\) are constant coefficients. (The assumption here spoken of is evidently equivalent to that made by Möbius.) Hamilton instances the function

\[ e^{-x^2} \]

as incapable of being put in the above form; for, could it be so assumed, (\(a\) denoting the least exponent,\(^1\)) we should have

\[ x^{-a}e^{-x^2} = A + Bx^{\beta-a} + Cx^{\gamma-a} \ldots . \]

which is impossible, since, when \(x = 0\), the first member becomes zero, while the second member takes the finite value \(A\). To show that the first member becomes zero, he considers the value of its reciprocal

\[ x^{-a}e^{-x^2} \]

when \(x = 0\); or putting \(x = 1 + y\) of \(y^{-a}e^{-y^2}\) when \(y\) is infinite. Expanding we have

\[ y^{-a}e^{-y^2} = y^{-a}\left(1 + y^2 + \frac{y^4}{2} + \frac{y^6}{3} + \&c.\right) \]

of which all the terms are positive, and after a limited number all are infinite when \(y\) is infinite; therefore

\[ x^{-a}e^{-x^2} \]

is infinit when \(x = 0\), and its reciprocal is zero when \(x = 0\).

The above demonstration would apply equally well to the function \(e^{-(1+x^2)}\), and this suggested to the author of No. 25 an example in which \(X^Y\) is not unity when \(X\) and \(Y\) vanish together. Putting this function for \(X\) and \(x\) for \(Y\) he has \((e^{-(1+x^2)})^x\) of which the value is constant and equal to \(1 + x\).

\((e^{(1+x^2)})^x\) is likewise an example; since it may be regarded as taking either the form \(0^0\), when \(x\) approaches zero from the negative side, or the form \(\infty^0\), when \(x\) approaches zero from the positive side.

The other writer who answers Möbius cites the function \(1 + \log x\) which vanishes with \(x\), and yet cannot be assumed in the form \(Px^n\); for, could it be so assumed, we should have
\[
\frac{1}{p} = x^m \log x = \log x^m.
\]

The demonstration of Möbius applying to the expression
\[
x^m,
\]
we have when \( x = 0 \)
\[
\frac{1}{p} = \log 1 = 0,
\]
but by hypothesis \( p \) is a finite quantity. Making use of this function he constructs the example \( x^{\alpha + \frac{1}{\log x}} \) which takes the form \( 0^0 \) when \( x = 0 \), but it is equivalent to \( e^{\alpha + \log x} \), hence when \( x = 0 \) its value is \( e^\alpha \).

The condition under which a function may assume the form \( 0^0 \) with a value other than unity may be found as follows: Let \( u^v \) be the function, \( u \) and \( v \) vanishing simultaneously. If \( u \) vanish with a finite ratio to any finite power of \( v \), say \( v^m \), let
\[
\left[ \frac{u}{v^m} \right]_0 = m;
\]
then, since \( v^m \) vanishes simultaneously.
\[
\left[ u^v \right]_0 = m^0 v^m \left[ v^v \right]_0 = m^0.1^m = 1.
\]

But if there is no finite value of \( n \), we fail to establish the equation. Hence the only exceptional cases are those in which
\[
\left[ \frac{u}{v^m} \right]_0 = 0,
\]
for all finite values of \( n \).

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SOLUTIONS OF PROBLEMS IN NUMBER THREE.

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SOLUTIONS of problems in No. 3 have been received as follows:

From Marcus Baker, 111, 112, 113, 114 and 118; Dr. H. Eggers, 113; [Dr. Eggers also furnished an elegant solution of 109, which was overlooked in our notice of solutions of problems in No. 2;] E. S. Farrow, 111; Henry Gunther, 111, 112, 113, 114 and 118; Orra Garvin, 111; Flora Henderson, 111; William Hoover, 111, 112, 113, 114, 116 and 118; W. W. Johnson, 114 and 118; Christine Ladd, 111, 112, 114, and 118; Artemas Martin, 111, 113, 114 and 116; Dr. A. B. Nelson, 111, 113, 114 & 116; O. D. Oathout, 111, 114 and 116; Prof. J. Scheffer, 111, 112, 113, 114 and 118; Anna T. Snyder, 111 and 114; S. W. Salmon, 113, 116 and 118; E. B. Seitz, 116.

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111. "Given \( \frac{x + y}{x - y} + \frac{x - y}{x + y} = 3 \frac{1}{x} \ldots (1) \), \( x^2 + y^2 = 45, \ldots (2) \) to be solved without the use of an auxiliary unknown."
SOLUTION BY ARTEMAS MARTIN, ERIE, PA.

Clearing (1) of fractions we get
\[ 5(x^2 - y^2) = 3(x^2 + y^2), \quad 135 \text{ by (2)}. \]
\[ x^2 - y^2 = 27. \quad \text{..................................(3)} \]
From (2) and (3) we easily find
\[ x = 6, \quad y = 3. \]

112. "Wishing to know the height of a tower standing at the summit of a slope on the opposite side of the street, and not being able to leave my room, I measured the angles of elevation of the bottom and top of the tower 40°, 70°, respectively, and the angle of depression of the foot of the slope 40°; and a passer-by carried a tape-line across, giving me the distance from my point of observation to the foot of the slope, 50 feet. I knew the angle made by the face of the slope with the horizontal plane of the street to be 60°. From these data, I found the height of the tower, having given, log tan 20° equal 9.651066, log sin 50° = 9.884254, log 10 = 1, and log 171072 = 5.238188."

SOLUTION BY CHRISTINE LADD, UNION SPRINGS, N. Y.

Let \( A \) be the point of observation, [the reader can readily construct the figure from the description,] \( C \) the foot of the slope, \( B \) its top and \( T \) the top of the tower. The bisector \( BD \), of the angle \( ABC \), is perpendicular to \( AC \), for \( BDA = 180° - 80° - 10° = 90° \). \( AD = 25 \), \( AB = 25 \div \sin 10° \), and because \( ATB = 20° \) and \( TAB = 30° \),
\[ TB = \frac{25}{\sin 10°} \cdot \frac{\sin 30°}{\sin 20°} = 25(\cot 20° + \cot 10°), = 210.47 \text{ ft.} \]
a solution shorter than that indicated by the problem.

[No solution of this question, employing only the data named in its announcement, has been received, though several of the solutions received use apparently more simple data.]

113. "The sides of a plane triangle are in arithmetical progression, common difference \( d \), and the angle opposite the least side is one third of the angle opposite the greatest side; construct the triangle."

SOLUTION BY MARCUS BAKER, U. S. COAST SURVEY, WASHINGTON, D. C.

Let the sides of the triangle \( ABC \) be \( x - d \), \( x \) and \( x + d \), and the angles opposite \( x - d \) and \( x + d \), \( \theta \) and \( 3\theta \), respectively.
From $A$ lay off $CAD$ equal to $\theta$ and prolong $AD$ till it meets $BC$ produced in $D$. Then $BDA = BCA - CAD = 3\theta - \theta = 2\theta$ and $BAD = 2\theta$ by construction; therefore the triangle $ABD$ is isosceles and hence $CD = (x + d) - (x - d) = 2d$.

Because $AC$ is the bisector of $BAD$, $AD = \frac{2d(x + d)}{x - d}$.

Now $AB \times AD = AC^2 + BC \times CD$ or $2d(x + d)^2 + (x - d) = x^2 + 2d(x - d)$. Reducing, $2d[(x + d)^2 - (x - d)^2] = x^2(x - d)$ or $8d^2 = x(x - d)$, and this value of $x$ is readily constructed.

114. "Required the shortest proof of the Pons Asinorum. (Eucl. 47, I.)"

SOLUTION BY DR. A. B. NELSON, DANVILLE, KENTUCKY.

By Quaternions, denoting the sides of the triangle in order by the vectors $a, \beta, \gamma$, and $\alpha$ including the right angle, we have

$$a + \beta = -\gamma,$$

or $a^2 + 2a\beta + \beta^2 = \gamma^2$.

But, since $a$ and $\beta$ are perpendicular to each other, $Sa\beta = 0$. Hence, passing from vectors to lines, $a^2 + \beta^2 = c^2$. Q. E. D.

115. "Show, by a geometrical construction, that the results obtained by Miss Ladd and Prof. Chase, in their solutions of Prob. 106, are the same."

SOLUTION BY THE EDITOR.

Let the sides $AB$, $AC$ and $BC$ of the triangle $ABC$ be represented by $a$, $\beta$, and $\gamma$, respectively, and let the angle $BAC = \theta$.

Draw $CA'$ equal and parallel to $AB$, join $BA'$ and draw the diagonal $AA'$; then is $a + \beta = AA'$ and $a - \beta = \gamma = BC$.

Let a circle circumscribe the triangle $ABC$ and draw its diameter $AD$. Draw $AF$ and $CE$ respectively perpendicular to $BC$ and $AB$, and join $DB$, $DC$ and $DF$.

Because $ABD$, $ACD$ and $AFD$ are angles in a semicircle they are all right angles, and therefore $DB$ is parallel to $CE$ and $DF$ is parallel to $CB$. 
Draw BH parallel and equal to CD, and produce HB to meet DF produced in G; then is BG = BH = CD. And because BH is parallel to CD it is perpendicular to AC, and therefore the point H is at the intersection of the two perpendiculars AF and CE. Hence the triangles ABH and A'CD, having two sides in one equal and parallel to two sides in the other, are identical, and therefore DA' is equal and parallel to AH.

Because BC and DG are parallel the alternate angles CBD and GDB are equal; and because GDB = FDB = FAB, and CBD = CAD, \( \angle HAB = \angle DAC = \angle HAD = \angle GA'B; \ldots \angle CA'D = \angle GA'B. \)

Therefore the angle GA'D equals the angle CAE, and because AD is parallel to AH it is therefore perpendicular to DG and the triangles ACE and A'GD are similar. Hence, as DG = BC = \( a - \beta \) and \( \angle DA'G = \theta \), we have

\[
A'D = \frac{(a - \beta) \cos \theta}{\sin \theta},
\]

Prof. Chase's answer is therefore represented by AD, the diagonal of the parallelogram AA'DI, while Miss Ladd's answer is represented also by AD, the diameter of a circle which circumscribes the quaternion \( \beta \div a \).

116. "A sector less than a semicircle is cut at random from a given circle, and a circle inscribed in it. Find the average area of this inscribed circle."

**SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.**

Let O represent the center of the given circle, BOD the sector, and OA a radius of the given circle through M, the center of the inscribed circle; and let MN be a perpendicular on OD. Put OA = r, MA = MN = x, \( \angle BOD = \theta \). Then \( x = (r - x) \sin \frac{1}{2} \theta \), whence \( x = r \sin \frac{1}{2} \theta / (1 + \sin \frac{1}{2} \theta) \), and the required average

\[
= \frac{1}{\pi} \int_0^\pi x^2 d\theta = r^2 \int_0^\pi \left( 1 - \frac{1}{1 + \sin \frac{1}{2} \theta} \right)^2 d\theta
\]

\[
= r^2 \int_0^\pi \left[ 1 - \frac{1}{2} \sec^2 \frac{1}{2} (\pi - \theta) \right] \frac{d\theta}{2}
\]

\[
= r^2 \left[ \theta + 3 \tan \frac{1}{2} (\pi - \theta) - \frac{1}{2} \tan^3 \frac{1}{2} (\pi - \theta) \right]_0^\pi = r^2 (\pi - \frac{\pi}{3}).
\]

117. "Required the average area of all the acute-angled triangles that can be inscribed in a given ellipse."
SOLUTION BY R. J. ADOCK, MONMOUTH, ILLINOIS.

Let $a^2y^2 + b^2x^2 = a^2b^2$, be the equation of the ellipse. From any point $P$ in the first quadrant $BD$ of the ellipse, draw any chord $PR$, and from its extremities the perpendiculars $PM$ and $RL$. Upon $PR$ as a diameter describe a circle intersecting the ellipse at $S$ and $T$. Let the abscissas of the $P$'s $P$, $R$, $M$, $L$, $T$, $S$, be respectively $x'$, $x''$, $x_1$, $x_2$, $x_3$, $x_4$, and let the chord $PR = 2r$, $p$ and $p'$ being perpendiculars upon the chord from any points in the elliptic arcs $LT$ and $SM$, respectively, and $dz$, the elementary arc of the ellipse.

Then while $PR$ is fixed we shall have

$$r\int \frac{x_3}{x_2}pdz + r\int \frac{x_1}{x_4}p'dz$$

= the sum of the areas of all the acute angled triangles that can be described on it.

Next regarding $P$ as fixed and $R$ movable, and integrating, after multiplying the preceding sum, now variable, by $dz$, we have

$$\int^{x''}_{x'} r \left[ \int \frac{x_3}{x_2}pdz + \int \frac{x_1}{x_4}p'dz \right] dz$$

= the sum of the areas of all the acute angled triangles having a common vertex $P$.

Multiplying again by $dz$, and integrating the preceding integral, now variable in consequence of $P$ occupying all the different points in the quadrant $BD$, we get, finally,

$$4\int^{x''}_{x'} \left[ \int^{x''}_{x'} r \left( \int \frac{x_3}{x_2}pdz + \int \frac{x_1}{x_4}p'dz \right) dz \right] dz$$

= the sum of the areas of all the acute angled triangles that can be inscribed in the ellipse; which divided by

$$4\int^{x''}_{x'} \left[ \int^{x''}_{x'} \left( \int \frac{x_3}{x_2}dz + \int \frac{x_1}{x_4}dz \right) dz \right] dz,$$

the whole number of acute triangles, gives the average required.

It should be observed that for each of the above integrals, except the last which is confined to the quadrant $BD$, the sum of two is to be used, one for the positive values of $T$ and the other for the negative values.

118. "Find the general relation which exists between the four sides and the two diagonals of any quadrilateral. Consider particularly the case when the opposite sides are equal."
SOLUTION BY PROF. J. SCHEFFER, COLLEGE OF ST. JAMES, MD.

Let a, b, c, d represent the four sides of the quadrilateral, and f and g the two diagonals; also let x, y, z, t represent the four segments of the diagonals and \( \alpha \) the angle which these diagonals make with each other. Then we can easily form the following equations;

\[
\begin{align*}
a^2 &= x^2 + t^2 + 2xt\cos \alpha, \quad \ldots \quad (1) \\
b^2 &= x^2 + z^2 - 2xz\cos \alpha, \quad \ldots \quad (2) \\
c^2 &= y^2 + z^2 + 2yz\cos \alpha \quad \ldots \quad (3) \\
d^2 &= y^2 + t^2 - 2yt\cos \alpha \quad \ldots \quad (4)
\end{align*}
\]

Adding (1) and (3), and (2) and (4), and subtracting the latter sum from the former we obtain,

\[
a^2 + c^2 - b^2 - d^2 = 2(xz + yz + xt + gt)\cos \alpha;
\]

but

\[
xz + yz + xt + gt = (x + y)(x + t) = fg;
\]

hence

\[
a^2 + c^2 - b^2 - d^2 = 2fg\cos \alpha,
\]

and

\[
sine\alpha = \frac{1}{2fg}\sqrt{4f^2g^2 - (a^2 + c^2 - b^2 - d^2)^2}.
\]

Denoting the area of the quadrilateral by \( A \), we have

\[
A = \frac{1}{2}fg\sin \alpha = \frac{1}{2}\sqrt{4f^2g^2 - (a^2 + c^2 - b^2 - d^2)^2},
\]

whence we get the relation

\[
4f^2g^2 = (a^2 + c^2 - b^2 - d^2)^2 + 16A^2.
\]

This formula enables us to find a diagonal if the four sides and the other diagonal are given, for the area can easily be expressed by the four sides and one diagonal.

In case of a parallelogram we have \( a = c, b = d \),

\[
\therefore \quad 16A^2 = 4(2a^2b^2 + 2a^2g^2 + 2b^2g^2 - a^4 - b^4 - g^4),
\]

consequently

\[
4f^2g^2 = 4(a^2 - b^2)^2 + 4(2a^2b^2 + 2a^2g^2 + 2b^2g^2 - a^4 - b^4 - g^4),
\]

or simplified,

\[
f^2 + g^2 = 2(a^2 + b^2): \text{ Hence—}
\]

In a parallelogram the sum of the squares of the two diagonals is equal to the sum of the squares of the four sides. This is a particular case of the more general theorem, first given by Euler, and called after him, "Euler's Theorem."—In any quadrilateral the sum of the squares of the diagonals is equal to the sum of the squares of the four sides increased by four times the square of the line connecting the middle points of the diagonals.

If \( \alpha = 90^\circ \), we have \( a^2 + c^2 = b^2 + d^2 \). And by substitution we easily obtain

\[
\left(\frac{f^2 + b^2 - c^2}{2f}\right)^2 + \left(\frac{g^2 - c^2 + d^2}{2g}\right)^2 = a^2.
\]
QUERY 1. "Which is the most effective; a brake applied at the top or the side of a car wheel in motion?"

[No answer to this query has been received. It is clear however, we think, that, so far as the retarding force of the brake is concerned, it is immaterial at what point of the periphery of the wheel the brake is applied. For, the force necessary to produce a given amount of friction between the brake and the wheel depends wholly upon the distance from the center of the wheel to the point of application of the brake, and as that distance is the same whether the brake be applied at the top or side of the wheel, it is clear that, so far as regards the retarding force of the brake, its position is immaterial, provided it acts on the periphery of the wheel.]

QUERY 2. "Can the equation $x^n + \sqrt{x} = a$, be solved?"

Answered by William Hoover, O. D. Oathout and Prof. Scheffer. Mr. Oathout represents the value of $x$ by a continued fraction. Mr. Hoover and Prof. Scheffer say it can not, in general terms, because it produces an equation of a degree denoted by $n^2$.

Prof. Ludwick submits the following as a substitute for his published solution of problem 105, which is defective.

Let $q =$ sum of combinations of $m$ things taken odd numbers at a time, $p = \text{even}$. Let $q + p = m + A + B + C + D + &c.$ then $q - p = m - A + B - C + D - &c.$

$\therefore (q + p) + (q - p) = 1 + \text{rem.}, (a) \text{ suppose,}$

or $q + p = q - p + ag - ap.$

$\therefore q = p + 2p + a. \therefore q > p.$

Hence the probability is in favor of drawing an odd number.

[Prof. A. B. Evans gave an elegant solution of this question and showed that the affirmation is not necessarily true unless the minimum number in a handful is 1 and the maximum number $n$, the whole number of shot in the bag.]

Note by Prof. A. Hall. — Mr. Ivory was not the discoverer of the property referred to by Mr. Siverly in Analyst, No. 3, p. 74. This was found by Jacobi, whose note is published in Poggendorf's Annalen, Vol. 33, p. 229.
SOLUTION OF THE PASTURAGE PROBLEM.

BY ARTEMAS MARTIN, ERIE, PA.

As it appears in Newton's Universal Arithmetic, Wilder's Edition, page 189, Problem XI, the general problem reads: "If the number of oxen $a$ eat up the meadow $b$ in the time $c$; and the number of oxen $d$ eat up as good a piece of pasture $e$ in the time $f$, and the grass grows uniformly; to find how many oxen will eat up the like pasture $g$ in the time $h$.

If $b$ acres of pasture, and the grass that grows on it during that time, keep $a$ oxen $c$ weeks, one ox eats $b \div a$ of an acre, and its growth during that time; and he eats $1 \div c$ as much in one week, or $b \div ac$ of a: acre and $b \div a$ of what grows on one acre in one week, supposing a week to be the unit of time.

In the second case, $e$ acres pasture $d$ oxen $f$ weeks, and one ox eats $e \div d$ of an acre, and what grows on it during that time; and in one week he eats $1 \div f$ as much, or $e \div df$ of an acre and $e \div d$ of what grows on an acre during a week.

Now, since one ox eats the same quantity of grass in one week in each case, therefore $e \div d - b \div a = (ae - bd) \div ad$ of the growth of one acre during one week is equal to $b \div ac - e \div df = \frac{bd - ace}{acdf}$ of an acre; and

$$\frac{bd - ace}{acdf} + \frac{ae - bd}{ad} = \frac{bd - ace}{acf(ae - bd)}$$

of an acre, is the growth of an acre during one week.

$$\frac{b}{ac} + \frac{b}{a} \cdot \frac{bd - ace}{acf(ae - bd)} = \frac{be(f - c)}{acf(ae - bd)}$$

the part of the original quantity on one acre which each ox eats in one week.

$$\frac{be(f - c)}{acf(ae - bd)} = \text{quantity of grass, in acres, one ox will eat in } h \text{ weeks.}$$

$$g + \frac{gh(bdf - ace)}{cf(ae - bd)} = \text{quantity of grass, in acres, to be eaten from } g \text{ acres of pasture in } h \text{ weeks; and}$$

$$\left[ g + \frac{gh(bdf - ace)}{cf(ae - bd)} \right] \div \frac{be(f - c)}{cf(ae - bd)} = \frac{c(af - bd) + gh(bdf - ace)}{be(f - c)}$$

is the number of oxen required to eat it, which agrees with Newton's result.

The foregoing method of solution is believed to be original. I published a numerical solution by this method in the April No. of Our Schoolday Visitor for 1868, page 109, and have since published similar solutions in several other periodicals.
REVISED SOLUTION OF PROBLEM EIGHTY NINE.

BY R. J. ADCOCK, MONMOUTH, ILLINOIS.

Problem.—"A sphere, radius $r$, rolls down the surface of another sphere of the same material, radius $R$, placed on a horizontal plane. The surfaces of both spheres and plane are rough enough to secure perfect rolling. Determine the motion of the spheres, the point of separation and the equation of the curve described by the center of the upper sphere."

Let $x$, $y$, be the coordinates, referred to the axes $AX$, $AY$, of the centre $G$ of the upper sphere at time $t$ after motion begins, $\phi =$ the angle $GCX$ which the line of centres $CG$ makes with the axis of $X$, $a =$ initial value of $\phi$, $a =$ initial value of $x$; $\theta_1 = a =$ angular rotation of lower sphere about its centre, $R(\theta_1 - a) =$ horizontal distance moved by $C$ the centre of the lower sphere; then

\begin{align*}
x &= a + (r + R)(\cos \phi - \cos a) - R(\theta_1 - a), \ldots \ldots (1) \\
y &= (r + R)\sin \phi, \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (2) \\
\frac{R(\phi - a) - R(\theta_1 - a)}{r} + \frac{\phi - a}{r} &= (r + R)\psi - R\theta_1 - a = \theta_2, \ldots (3)
\end{align*}

= angular rotation of upper sphere,

\begin{align*}
dx^2 + dy^2 &= R^2d\theta_1^2 + 2R(r + R)\sin \phi \, d\phi \, d\theta_1 + (r + R)^2d\psi^2 \ldots (4)
\end{align*}

There being no loss of vis viva in perfect rolling,

\begin{align*}
2m'g(r + R)(\sin a - \sin \phi) &= m'r^2\frac{dx^2}{da^2} + \frac{2}{5}m'r^2\frac{d\theta_1^2}{da^2} + \frac{7}{5}mR^2\frac{d\theta_1^2}{da^2} \\
&= \frac{7}{5}(m + m')R^2\frac{dx^2}{da^2} + \frac{7}{5}m'(r + R)^2\frac{d\psi^2}{da^2} + 2m'R[r + R](\sin \phi - \frac{2}{5})\frac{d\phi \, d\theta_1}{da^2} \\
&\ldots \ldots (5)
\end{align*}

Let $F =$ the resultant action between the spheres at the point of contact $H$, $\phi =$ the angle $CHK$ which its direction makes with the normal $HC$, then the equations of motion of the upper sphere are

\begin{align*}
F \cos (\phi - \phi) &= m\frac{d^2x}{da^2}, \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (6) \\
F \sin (\phi - \phi) - m'g &= \frac{d^2y}{da^2}, \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (7)
\end{align*}
\[ \frac{F \sin \varphi}{\frac{1}{2} m' r^2} = \frac{F \sin \varphi}{\frac{1}{2} m' r} = \frac{d^2 \theta}{dx}, \ldots \ldots \ldots \ldots \ldots \ldots \quad (8) \]

and for rotation of lower sphere about lower point of contact \( B \),

\[ \frac{F \cos (\varphi - \varphi) \times R (1 + \sin \psi) - F \sin (\varphi - \varphi) \times R \cos \varphi}{\frac{1}{2} m R^2} = \frac{F \times \cos (\varphi - \varphi) + \sin \varphi}{\frac{1}{2} m R} = \frac{d^2 \theta}{dx}, \ldots \ldots \ldots \quad (9) \]

Eliminating between (6), (8), (9),

\[ m' \frac{d^2 x}{dt^2} + 2 \frac{m'}{5} m' r \frac{d^2 \theta}{dx} = \frac{7}{5} m R \frac{d^2 \theta}{dx}, \ldots \ldots \ldots \ldots \ldots \ldots \quad (10) \]

Integrating and observing that the velocities begin at the same time,

\[ m' \frac{dx}{dt} + 2 \frac{m'}{5} m' r \frac{d \theta}{dx} = \frac{7}{5} m R \frac{d \theta}{dx}, \ldots \ldots \ldots \ldots \ldots \ldots \quad (11) \]

Eliminating by (1) and (3)

\[ \frac{d \theta}{dx} = \frac{m' (r + R) \left( \frac{1}{2} - \sin \varphi \right) \frac{dy}{dx}}{\frac{1}{2} R (m + m')} \ldots \ldots \ldots \ldots \ldots \ldots \quad (12) \]

Integrating between limits,

\[ \theta \left| _{1} = \alpha = \left[ \frac{3}{2} (\varphi - \alpha) + \cos \varphi - \cos \alpha \right] m (r + R), \ldots \ldots \ldots \ldots \ldots \quad (13) \]

Hence by (1) and (2)

\[ x = \alpha + (r + R) (\cos \varphi - \cos \alpha) - m' (r + R) \left[ \frac{3}{2} (\varphi - \alpha) + \cos \varphi - \cos \alpha \right] \frac{1}{2} (m + m') \]

\[ = \alpha + \left[ \frac{7 (m + m') - 5 m'}{7 (m + m')} \right] (r + R) (\cos \varphi - \cos \alpha) - 2 m' (r + R) (\varphi - \alpha) \]

\[ = \left( 7 m + 2 m' \right) \left\{ \left[ (r + R)^2 - y^2 \right] - (r + R) \cos \alpha \right\} \]

\[ - 2 m' (r + R) \left( \sin^{-1} \frac{y}{r + R} - \alpha \right), \ldots \ldots \ldots \ldots \ldots \ldots \quad (14) \]

which is the equation of the required curve.

Changing signs of (14), it is seen that as \( y \) decreases, \( \alpha - x \) increases, therefore the motion of the upper sphere is toward the origin \( A \) as represented in the figure; and from (12) and (13) the motion of the lower sphere is, for values of \( \alpha \) not much greater than \( \frac{1}{2} \pi \), at first, from the origin \( A \), until \( \sin \varphi = \frac{1}{2} \), when it comes to rest, where its motion is reversed, and returns to its original position when \( \frac{3}{2} (\varphi - \alpha) + \cos \varphi - \cos \alpha = 0 \).

At the time of separation, since the component of \( F \) normal to the two surfaces equals zero, equation (9) becomes
\[ \frac{F \sin \varphi (1 + \sin \varphi)}{\frac{4}{5} m R} = \frac{d^2 \theta}{d^2 t}, \quad \ldots \ldots . \quad (15) \]

hence by (8)
\[ \frac{7}{5} m R \frac{d^2 \varphi}{d^2 t} = \frac{2}{5} m' r \frac{d^2 \theta}{d^2 t} = \frac{2}{5} m' \left[ (r + R) \frac{d^2 \psi}{d^2 t} - R \frac{d^2 \theta}{d^2 t} \right], \]

from which
\[ \frac{d^2 \theta}{d^2 t} = \frac{2 m'(r + R) (1 + \sin \psi) \frac{d \psi}{d^2 t}}{R [7 m + 2 m'(1 + \sin \psi)]}, \ldots \ldots . \quad (16) \]

From (16) I conclude, that at the point of separation the tendency of the forces acting on the lower sphere is to turn it in the same direction as the rotation of the centre of the upper about the lower.

From (16) and (12)
\[ \frac{d^2 \varphi}{d^2 t} = \frac{2 m'(r + R) (1 + \sin \psi)}{R [7 m + 2 m'(1 + \sin \psi)]} \cdot \frac{m'(r + R) \left[ (\frac{4}{5} - \sin \varphi) \frac{d^2 \psi}{d^2 t} - \cos \varphi \frac{d \psi}{d^2 t} \right]}{\frac{2}{5} R (m + m')}, \]

hence
\[ \frac{d^2 \psi}{d^2 t} = \frac{7 m + 2 m'(1 + \sin \psi) \cos \varphi + \frac{2}{5} (m + m') (1 + \sin \psi) \times \frac{d \psi}{d^2 t}}{(\frac{4}{5} - \sin \varphi) [7 m + 2 m'(1 + \sin \psi)]}, \quad (17) \]

And from (5) and (12),
\[ \frac{d^2 \psi}{d^2 t} = \frac{\frac{4}{5} g (m + m') (\sin \alpha - \sin \psi)}{(r + R) [\frac{4}{5} (m + m') - m'(\frac{4}{5} - \sin \varphi)]}, \quad \ldots \ldots , \quad (18) \]

Differentiating (18),
\[ \frac{d^2 \psi}{d^2 t} = - \frac{\frac{4}{5} (m + m') - m'(\frac{4}{5} - \sin \varphi) + 2 m'(\sin \alpha - \sin \psi) (\frac{2}{5} - \sin \varphi)}{[\frac{4}{5} (m + m') - m'(\frac{2}{5} - \sin \varphi)]} \times \frac{\frac{4}{5} g (m + m')}{r + R} \times \cos \varphi; \]

from which by (17) and (18)
\[ \frac{7 m + 2 m'(1 + \sin \psi) \cos \varphi + \frac{2}{5} (m + m') (1 + \cos \varphi) \times (\sin \alpha - \sin \psi)}{(\frac{4}{5} - \sin \varphi) [7 m + 2 m'(1 + \sin \psi)]} \]

\[ = - \cos \varphi \times \frac{\frac{4}{5} (m + m') - m'(\frac{4}{5} - \sin \varphi) + 2 m'(\sin \alpha - \sin \varphi) (\frac{2}{5} - \sin \varphi)}{\frac{4}{5} (m + m') - m'(\frac{2}{5} - \sin \varphi)^2}, \]

which is the equation for the point of separation.

[The foregoing revised solution of problem 89 by Mr. Adcock, is published by his request and entirely at his expense; an extra half sheet having been added to make room for it.]
PROBLEMS.

119. SELECTED, BY DR. WM. HILLHOUSE, NEW HAVEN, CONN.—If from one of the extremities, B, of the diameter, AB, of a given circle, any chord, BD, be drawn, and from the other extremity, A, and the centre, C, two lines, AE, CE, be drawn to any point, E, in the circumference, cutting said chord in the points F and G; then, GF : GD :: (BF)² : (BA)².
Required the demonstraton.

120. BY DR. N. R. OLIVER, LONDON ONTARIO.—Given
\[
\begin{align*}
(x^2 + xy + y^2)^{1\over 3} &= m, \\
(x^2 - xy + y^2)^{1\over 3} &= n.
\end{align*}
\]
To find x and y.

121 By PROF. A HALL.—In a spherical triangle are given the sum of each angle and the side opposite, to solve the triangle.

122. BY PROF. H. T. J. LUDWICK.—It has been cloudy during the last seven days; what is the probability that it will be cloudy to-morrow?

122. BY DR. H. EGGERS, MILWAUKEE, WISCONSIN.—To inscribe a square in a given quadrilateral.

123. BY MARCUS BAKER, U. S. COAST SURVEY.—Solve the equation \(\sqrt[n]{x} = a\) and determine what values of a give real roots.

124. BY PROF. W. W. HENDRICKSON U. S. N. ACADEMY.—A radius is drawn in the circle \(x^2 + y^2 = a^2\) and from its extremity an ordinate. From the foot of the ordinate a line is drawn perpendicular to the radius. Find and discuss the envelop of this last line.

BOOK NOTICE.


This handsome little book will well repay a perusal by anyone who has calculations to make in which he deals with quantities whose approximate values only are known.

The chapter treating of logarithms and trigonometric functions is especially worthy of notice and should be read by every student of trigonometry.

Correction. On page 93, lines 11 and 12, invert the two signs of inequality.
NEW INVESTIGATION OF THE LAW OF ERRORS OF OBSERVATION.

BY CHAS. H. KUMMEL, ASSISTANT U. S. LAKE SURVEY, DETROIT, MICH.

1. ERRORS of observation being freed of constant and regular errors can be only subject to the laws of probability, and we may conceive an error $d$ to be the result of two opposing influences; one tending to make the observation greater and the other smaller than the true result. If the increasing influence is the greatest a positive error is the result, and vice versa. Now conceive these influences to be made up from $2\Omega = \infty$ equal element errors $\pm i$, some positive and some negative and let $(\Omega + v)$ times $+ i$ and $(\Omega - v)$ times $- i$ occur simultaneously to produce an error $d$ then we have

\[
(\Omega + v)i - (\Omega - v)i = 2vi = d.
\]

To produce an error $d + dA$ we must suppose one more $+ i$ and one less $- i$ to preserve the constant total number $2\Omega$, that is

\[
(\Omega + v + 1)i - (\Omega - v - 1)i = 2(v + 1)i = d + dA.
\]

We have then

\[
i = \frac{1}{2}dA,
\]

\[
vdA = d.
\]

We have here two events of equal probability, viz., $+ i$ and $- i$ and their simple probability is $= \frac{1}{2}$. The terms of the development

\[
\left(\frac{1}{2} + \frac{1}{2}\right)^{2\Omega} = 1
\]

will give then the probability of any number of $+ i$ and the remaining number of $- i$ to occur simultaneously. The middle term, which is the largest, gives the probability of $+ \Omega i$ and $- \Omega i$ occurring simultaneously or what is the same thing that of the error $0$. If we denote this by $\varphi_0$ and
so on the following terms on both sides by increasing integer subscripts, then $\varphi$, gives the probability of $(\Omega + v)i$ and $-(\Omega - v)i$ or by (1) that of the error $\Delta$. We have then

$$\varphi_0 = 2^{-2\Omega} \cdot \frac{2\Omega}{1} \cdot \frac{2\Omega - 1}{2} \cdots \frac{\Omega + 1}{\Omega}$$

= probability of error $0 = (\Omega - \Omega)i$,

$$\varphi_1 = 2^{-2\Omega} \cdot \frac{2\Omega}{1} \cdot \frac{2\Omega - 1}{2} \cdots \frac{\Omega + 2}{\Omega - 1}$$

= probability of error $\Delta = (\Omega + 1)i - (\Omega - 1)i$,

$$\cdots$$

$$\varphi_\Omega = 2^{-2\Omega} \cdot \frac{2\Omega}{1} \cdot \frac{2\Omega - 1}{2} \cdots \frac{\Omega + v}{\Omega - v}$$

= probability of error $\Delta = (\Omega + v)i - (\Omega - v)i$,

$$\varphi_{v+1} = 2^{-2\Omega} \cdot \frac{2\Omega}{1} \cdot \frac{2\Omega - 1}{2} \cdots \frac{\Omega + v + 2}{\Omega - v - 1}$$

= probability of error $\Delta + \Delta = (\Omega + v + 1)i - (\Omega - v - 1)i$,

$$\cdots$$

$$\varphi_{\Omega-1} = 2^{-2\Omega} \cdot \frac{2\Omega}{1} = \text{probability of error} (\Omega - 1)i$$

$$\varphi_\Omega = 2^{-2\Omega} = \text{probability of error} \Omega \Delta = 2\Omega i - 0.i = \infty.$$}

These probabilities evidently form a continuous function. Denoting the general term $\varphi_v$ by $\varphi$ then $\varphi_{v+1} = \varphi + d\varphi$, and dividing (8) by (7) we have

$$\frac{\text{probability of } \Delta + \Delta}{\text{probability of } \varphi} = \frac{\varphi_{v+1}}{\varphi} = \frac{\varphi + d\varphi}{\varphi} = \frac{\Omega - v}{\Omega + v + 1}$$

$$\cdots$$

$$\frac{d\varphi}{\varphi} = - \frac{2v + 1}{\Omega + v + 1} = - \frac{2v \Delta + \Delta}{\Omega \Delta + v \Delta + \Delta}$$

(10)

and at the limit, since $\Omega \Delta = \infty$ by (9),

$$\frac{d\varphi}{\varphi} = - \frac{2 \Delta + \Delta}{\Omega \Delta + \Delta + \Delta}$$

(11)

$\Omega \Delta$ is evidently a finite constant. Since $\Omega \Delta = 4\Omega i = 2 \times 2\Omega i$, it is double the sum of the squares of the element errors $\pm i$, and placing this quantity, viz.,

$$2\Omega^2 = \frac{1}{2} \Omega \Delta \Delta = \varepsilon^2$$

we have

$$\frac{d\varphi}{\varphi} = - \frac{\Delta \Delta}{\varepsilon^2}.$$
Integrating we obtain \( \log \varphi = \log C - A^2 + 2s^2 \). If \( A = 0 \) then \( \log \varphi \)
= \( \log C \), hence

\[
\varphi = \varphi_0 e^{-\frac{\Delta^2}{2s^2}}.
\]

(14)

We can determine \( \varphi_0 \) from (6) for we have

\[
\varphi_0 = 2^{-2n} \frac{2 \Omega \cdot \Omega - 1}{1} \frac{2}{2} \cdots \frac{\Omega + 1}{\Omega} \quad (\Omega = \infty)
\]

\[
= 2^{-2n} \frac{1 \cdot 2 \cdot 3 \cdots 2 \Omega}{1 \cdot 2 \cdot 3 \cdots \Omega} \quad (\Omega = \infty)
\]

\[
= 2^{-n} \frac{1 \cdot 3 \cdot 5 \cdots (2 \Omega - 1)}{1 \cdot 2 \cdot 3 \cdots \Omega} \quad (\Omega = \infty)
\]

\[
\varphi_0 = \frac{1 \cdot 3 \cdot 5 \cdots (2 \Omega - 1)}{2 \cdot 4 \cdot 6 \cdots 2 \Omega}. \quad (\Omega = \infty)
\]

(15)

By Wallis' theorem

\[
\frac{\pi}{2} = \frac{2 \cdot 4 \cdot 6 \cdots (2n - 2)2n}{1 \cdot 3 \cdot 5 \cdots (2n - 1)2n}; \quad (n = \infty)
\]

\[
\therefore \quad (\Omega \pi)^n = \frac{2 \cdot 4 \cdot 6 \cdots 2n}{1 \cdot 3 \cdot 5 \cdots (2n - 1)}; \quad (n = \infty)
\]

(16)

\[ \varphi_0 = (\Omega \pi)^{-n}. \quad (\Omega = \infty) \]

This remarkable result proves that the probability to commit no error at all
is an absolute constant.

We have by (12)

\[
\Omega = \frac{2s^2}{dA^2};
\]

(17)

therefore also

\[
\varphi_0 = \frac{dA}{\sqrt{2\pi}}.
\]

(18)

and

\[
\varphi = \frac{dA}{\sqrt{2\pi}} e^{-\frac{\Delta^2}{2s^2}}.
\]

(19)

Let \( [\varphi]_a^b \) denote the probability of an error to lie between \( a \) and \( b \), then

\[
[\varphi]_a^b = \int_a^b \frac{dA}{\sqrt{2\pi}} e^{-\frac{\Delta^2}{2s^2}}.
\]

(20)

The sum of all the probabilities being certainty = 1, we must have

\[
[\varphi]_{-\infty}^{+\infty} = \int_{-\infty}^{+\infty} \frac{dA}{\sqrt{2\pi}} e^{-\frac{\Delta^2}{2s^2}} = 1;
\]

(21)

and this is the case since by Laplace's integral

\[
\int_{-\infty}^{+\infty} dx e^{-x^2} = \sqrt{\pi}.
\]

(22)
2. Supposing now an ideal case in which all the possible errors, viz., 0, \(dA, 2dA, \ldots, dA, dA + dA, \ldots\) \((Q - 1)dA, \Omega dA\) occur positive and negative exactly in proportion to their probabilities as given by (19). Let \(m\) be the total number of errors then we have

\[
m\varphi = \frac{mdA}{\varepsilon \sqrt{2\pi}} e^{-\frac{\Delta^2}{2\varepsilon^2}}
\]

= number of times the error \(d\) should occur and

\[
m\varphi d^2 = \frac{mdA}{\varepsilon \sqrt{2\pi}} d^2 e^{-\frac{\Delta^2}{2\varepsilon^2}}
\]

= sum of the squares of the errors. Following the usual Gaussian notation for sums of similarly formed quantities according to which, for inst.,

\[
a_1 + a_2 + \ldots + a_m = [a],
\]

\[
a_1^2 + a_2^2 + \ldots + a_m^2 = [a^2],
\]

\[
a_1b_1 + a_2b_2 + \ldots + a_mb_m = [ab], \text{ etc.,}
\]

we have the sum of the squares of all the errors in our ideal case

\[
[d^f] = m \int_{-\infty}^{+\infty} dA e^{-\frac{\Delta^2}{2\varepsilon^2}}
\]

\[
= \frac{m \varepsilon}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dA e^{-\frac{\Delta^2}{2\varepsilon^2}}
\]

\[
= \frac{m \varepsilon}{\sqrt{2\pi}} [\Delta e^{-\frac{\Delta^2}{2\varepsilon^2}}]_{-\infty}^{+\infty} + \frac{m \varepsilon}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dA e^{-\frac{\Delta^2}{2\varepsilon^2}}
\]

\[
= m \varepsilon^2.
\]

(23)

\[
\varepsilon = \sqrt{\left(\frac{[d^f]}{m}\right)}.
\]

There can be no objection to use this formula for a finite number of errors provided we admit that in the long run the law of frequency of errors which we have assumed in this ideal case to be strictly fulfilled, will hold also in experience the more perfectly the greater the number of observations.

As we shall see, the method which we are going to explain consists mainly in this: to determine the unknown quantities in such a way that the system of resulting errors \(A_1, A_2, \ldots, A_m\) resembles as much as possible the above ideal case, and this must bring us the nearer to truth the more regular the law of error has been followed.

The constant \(\varepsilon\) found by (23) is called the mean error of \(A_1, A_2, \ldots, A_m\) and denoting it by \(\varepsilon_\Delta\) we have

\[
(23') \quad \varepsilon_\Delta = \sqrt{\left(\frac{A_1^2 + A_2^2 + \ldots + A_m^2}{m}\right)} = \sqrt{\left(\frac{[d^f]}{m}\right)},
\]
and this quantity is to be distinguished from the mean error $\varepsilon$ of the observations which slightly exceeds $\varepsilon \Delta$ since it includes the uncertainty of the unknown quantities themselves. It is plain from all this that $\varepsilon$ the mean error of observation characterizes a class of observations, and this with regard to precision is plain from the following: If $\varepsilon'$ is the mean error of another system we have the probability of an error $\Delta'$ by (14)

$$\varphi' = \varphi_0 e^{-\frac{\Delta'^2}{2\varepsilon'^2}}.$$

If an error $\Delta$ in the first system is to have the same probability as an error $\Delta'$ in the second then must

$$\frac{\Delta}{\varepsilon \sqrt{2}} = \frac{\Delta'}{\varepsilon' \sqrt{2}}.$$

If then $\varepsilon' > \varepsilon$ then also $\Delta' > \Delta$, that is, greater errors in the second system have the same probability as smaller ones in the first.

Gauss puts

$$(24) \quad h = \frac{1}{\varepsilon \sqrt{2}}$$

and calls $h$ the measure of precision, which is the greater the lesser the class of observations.

3. The most usual method to compare the precision of different systems is by means of the probable error. This is a quantity which stands in the middle of the series of errors arranged according to their magnitude. We have then the probability to commit an error less than the probable error equal to that to commit an error greater, or, denoting it by $r$, we have regarding (20) and (21),

$$(25) \quad \int_{-\infty}^{\infty} \frac{d\Delta}{\varepsilon \sqrt{2\pi}} e^{-\frac{\Delta^2}{2\varepsilon^2}} = \int_{-\infty}^{\infty} \frac{d\Delta}{\varepsilon' \sqrt{2\pi}} e^{-\frac{\Delta^2}{2\varepsilon'^2}} = \frac{1}{4}.$$

Place

$$(26) \quad \Delta = \varepsilon \sqrt{2} = h \Delta = \theta,$$

then (25) becomes

$$(27) \quad \frac{1}{\sqrt{\pi}} \int_{0}^{\rho} d\theta e^{-\frac{\theta^2}{\varepsilon'^2}} = \frac{1}{\sqrt{\pi}} \int_{0}^{\rho} d\theta e^{-\theta^2} = \frac{1}{4}.$$

where we have put

$$(28) \quad r = \varepsilon \sqrt{2} = \rho.$$

If we solve (27) for $\rho$, which is most readily done by means of tables of the definite integral

$$\int_{0}^{\theta} d\theta e^{\theta^2},$$

we find
\[ \rho = 0.47694, \]

hence by (28)
\[ r = \epsilon \rho \varphi^2 = 0.6745 \epsilon. \]

This formula gives the relation between mean and probable error. We have then the following relation between mean error, probable error and precision
\[ h = \frac{1}{\epsilon \varphi^2} = \frac{\rho}{r}. \]

and the probability of an error \( A \) is
\[ \varphi = \frac{d\Delta}{\epsilon \sqrt{2\pi}} e^{-\Delta^2/2\epsilon^2} = \frac{h d\Delta}{\sqrt{\pi}} e^{-h^2 \Delta^2} = \frac{\rho d\Delta}{r \sqrt{\pi}} e^{-\rho^2 \Delta^2}. \]

4. The probable error is highly significant in its relation to a determined quantity. It is then an even wager that the error of that determination exceeds the probable error as that it is smaller. It is usually placed with a double sign after the quantity. Thus the equation
\[ x_1 = a_1 \pm r_1 \]

signifies that the most probable value of the unknown quantity \( x_1 \) has been found \( a_1 \) with such an uncertainty that the actual error of \( a_1 \) is with the same probability greater as it is less than \( r_1 \). The probability of any value to be \( = x_1 \) is therefore
\[ \varphi(x_1) = \frac{\rho dx_1}{r_1 \sqrt{\pi}} e^{-\frac{\rho^2}{r_1^2}(x_1-a_1)^2}. \]

Similarly we have if \( x_2 = a_2 \pm r_2 \)
\[ \varphi(x_2) = \frac{\rho dx_2}{r_2 \sqrt{\pi}} e^{-\frac{\rho^2}{r_2^2}(x_2-a_2)^2}. \]

If now
\[ X = a_1 x_1 + a_2 x_2 \]

and we require the most probable value and probability of \( x \) we cannot assume the first to be \( a_1 a_1 + a_2 a_2 \), at least not without proof. As this will be furnished as soon as we know the probability of \( X \) we shall determine this probability.

We evidently pass through all imaginable values for \( X \) if we combine in (34) any value of \( x_1 \) with any value of \( x_2 \); hence the probability of any special value of \( X \) is the compound probability
\[ \frac{\rho^2 dx_1 dx_2}{r_1^2 r_2^2 \pi} e^{-\frac{\rho^2}{r_1^2}(x_1-a_1)^2} - \frac{\rho^2}{r_2^2}(x_2-a_2)^2 \]

after \( x_1 \) has passed from \( + \infty \) to \( - \infty \), or
\[ \varphi(X) = \frac{\rho^2 dx_2}{r_1^2 r_2^2 \pi} \int_{-\infty}^{+\infty} \frac{dx_1}{e^{-\frac{\rho^2}{r_1^2}(x_1-a_1)^2} - \frac{\rho^2}{r_2^2}(x_2-a_2)^2}. \]
By (34) we have

\[
(36) \quad x_2 = \frac{X - a_1 x_1}{a_2},
\]

\[
(37) \quad dx_2 = \frac{dX}{a_2},
\]

since \( x_2 \) must be independent of \( x_1 \) after the \( x_1 \) integrations. Introducing these values into (35) we have

\[
(38) \quad \varphi(X) = \frac{\rho^2 dX}{a_2 r_1 r_2 r_3} \int_{-\infty}^{+\infty} dx_1 e^{-\frac{\rho^2}{a_1} (x_1 - a_1)^2} - \frac{\rho^2}{a_2 r_3^2} (X - a_1 x_1 - a_2 a_2)^2.
\]

The exponent may be put in the form \(-Ax_1^2 + 2Bx_1 - C\), hence

\[
\varphi(X) = \frac{\rho^2 dX}{a_2 r_1 r_2 r_3} \int_{-\infty}^{+\infty} dx_1 e^{-A(x_1 - B/A)^2 + B^2/A - C}
\]

\[
= \frac{\rho^2 dX}{a_2 r_1 r_2 r_3} e^{-(C - B^2/A)/A} \int_{-\infty}^{+\infty} \frac{\pi}{A} \cdot \sqrt{\frac{x_1 - B}{A}}.
\]

But by comparison

\[
A = \frac{\rho^2}{r_1^2} + \frac{\rho^2}{r_2^2} a_2^2 = \frac{\rho^2}{r_1^2 r_2^2 a_2} \left[ a_1^2 r_1^2 + a_2^2 r_2^2 \right],
\]

\[
B = \frac{\rho^2}{r_1^2} a_1 + \frac{\rho^2}{r_2^2} a_2 (X - a_2 a_2) = \frac{\rho^2}{r_1^2 r_2^2 a_2} \left[ a_1^2 a_2^2 r_2^2 + (X - a_2 a_2)^2 r_1^2 \right]
\]

\[
C = \frac{\rho^2}{r_1^2} a_2^2 + \frac{\rho^2}{r_2^2} (X - a_2 a_2)^2 = \frac{\rho^2}{r_1^2 r_2^2 a_2} \left[ a_1^2 a_2^2 r_2^2 + (X - a_2 a_2)^2 r_1^2 \right]
\]

\[
\frac{B^2}{A} - C = -\frac{\rho^2}{a_1^2 r_1^2 + a_2^2 r_2^2} \left[ \frac{a_1^2 a_2^2 r_2^2 + (X - a_2 a_2)^2 r_1^2}{a_2^2 r_2^2} \right] - \frac{\rho^2}{a_1^2 r_1^2 + a_2^2 r_2^2} \left[ X - a_1 a_1 - a_2 a_2 \right]^2.
\]

With these values (39) becomes

\[
(40) \quad \varphi(X) = \frac{\rho dX}{V(a_1^2 r_1^2 + a_2^2 r_2^2)\pi} e^{-\frac{\rho^2}{a_1^2 r_1^2 + a_2^2 r_2^2} (X - a_1 a_1 - a_2 a_2)^2}.
\]

The most probable value of \( X \) is therefore

\[
(41) \quad X_0 = a_1 a_1 + a_2 a_2,
\]

and its probable error

\[
(42) \quad R = \sqrt{a_1^2 r_1^2 + a_2^2 r_2^2},
\]

or the complete value

\[
(43) \quad X = a_1 a_1 + a_2 a_2 \pm \sqrt{a_1^2 r_1^2 + a_2^2 r_2^2}.
\]
If more generally

\[(34') \quad X = a_1 x_1 + a_2 x_2 + \ldots + a_m x_m = [a x],\]

where \(x_1 = a_1 \pm r_1, x_2 = a_2 \pm r_2, \ldots, x_m = a_m \pm r_m,\)

then we have by composition

\[(40') \quad \varphi(X) = \frac{\rho dX}{\sqrt{[(a^2)^2 + a^2]}} \exp\left(-\frac{\rho^2}{a^2} \frac{x - [a x]^2}{[a^2 x^2]^2}\right),\]

\[(43') \quad X = [a a] \pm \sqrt{[(a^2)^2 + a^2]}.\]

If

\[(34'') \quad X = f(x_1, x_2, \ldots, x_m),\]

the above integration cannot be effected but an approximate solution can be given in that case which is the nearer the perfect the smaller the probable errors \(r_1, r_2, \ldots, r_m.\) We have by Taylor's theorem, neglecting higher powers of increments \(\Delta x_1, \Delta x_2, \ldots, \Delta x_m\)

\[(34''') \quad X = f(a_1, a_2, \ldots, a_m) + f'(a_1) \Delta x_1 + f'(a_2) \Delta x_2 + \ldots + f'(a_m) \Delta x_m.\]

Within the range of \(\Delta x_1, \Delta x_2, \ldots, \Delta x_m,\) for which this form is exact enough, \(X\) is of the form \((34').\) In the integration however these increments have to pass from \(+ \infty\) to \(- \infty.\) If the probable errors are small this will make no sensible difference since the integral

\[\int -\infty^\infty \frac{\rho d\Delta}{\sqrt{\rho^2 + \pi}} \exp\left(-\frac{\rho^2}{\pi} \Delta^2\right)\]

approaches 0 the more rapidly the smaller \(\rho.\) This circumstance admits to a certain extent the treating of \(X\) as a linear function of \(\Delta x_1, \Delta x_2, \ldots, \Delta x_m\)

and we have

\[(40'') \quad \varphi(X) = \frac{\rho dX}{\sqrt{[f'(a)^2 + a^2]}} \exp\left(-\frac{\rho^2}{[f'(a)^2 + a^2]} [X - f'(a_1, a_2, \ldots)^2]\right),\]

\[(43'') \quad X = f(a_1, a_2, \ldots, a_m) \pm \sqrt{[f'(a)^2 + a^2]}.\]

(To be continued.)

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**NOTE BY S. W. SALMON.**—In the note on Differential Calculus (p. 14), I wrote \((y - u') / (u - u') = 1.\) This needs to be proved. If the rate of motion of the point \(B\) is increasing, just before \(x = x',\) \((y - u') / (u - u')\) is greater than 1, and just after \(x = x',\) it is less than 1; therefore when \(x = x',\)

\[(y - u') / (u - u') = 1.\] If \(B's\) rate is decreasing, it may be proved in a similar manner that \((y - u') / (u - u') = 1.\)
SOLUTION OF THE GENERAL EQUATION OF THE FIFTH DEGREE.

BY ADOLPH VON DER SCHULENBURG.

"IT STILL MOVES!"*

TRANSLATED BY DR. A. B. NELSON, PROFESSOR OF MATHEMATICS IN CENTRE COLLEGE, DANVILLE, KENTUCKY.†

Since I can thus give only a fragment (of my larger work) which is indeed a finished result but finds its complement in the part which must be withheld, I must apologize for external defects. It is to be hoped that, notwithstanding its brevity, it will not be wanting in clearness and intelligibility; and if this should anywhere still be the case, I can only answer with the apology that the addition of the explanatory operations is not possible under existing circumstances, because of their extent.

On account of this solution of the equation of the 5th degree, much sweat of thought has already flowed. From the old analysts who wished to find it to the later analysts who brought system into the attack upon this 'Veil of Isis,' it has been an object of algebraic effort; and with comparatively small result. Waring computed the product of differences, Euler knew the right position of the radicals in the roots, Lagrange the solution-sums and Olivier their application to the restoration of the coefficients of a given equation of the 5th degree; but still from the utilization of all these forces in the

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†My reason for offering this translation for publication is simply the fact that it seems to be unknown among mathematicians in this country that this solution has really been accomplished. As evidence of this generally prevailing impression, I cite the following extract from Educational Notes and Queries for February, 1875, Edited by Prof. W. D. Henkle, at Salem, Ohio: "It may not be known to the younger class of mathematicians that the impossibility of resolving algebraically general equations of the fifth or higher degrees has been rigorously demonstrated. This was first done by Abel. Wantzel's demonstration, which Serret says is simpler than Abel's, may be seen in Serret's Cours D'Algebre Superieure, 3rd Edition, 1866, Vol. 2."

Also, the following concluding paragraph from an article by Mr. G. W. Hill in the Analyser, Vol. II, No. 1: "If we should attempt to treat the general equation of the fifth degree in the preceding manner, we would be led to equations of higher degrees than the 5th, which must be regarded as a strong probable argument for the non-existence of an algebraic expression equivalent to the root of the general equation of this degree."

These statements, as far as I am aware, having passed unchallenged for over a year and a half, I conclude that the solution here given to the American public is not generally known. Translator.
solution of the problem, we were so distant that Abel in recent times could
bring forward his celebrated proof of the impossibility of this solution:
whose abundance of material, whose dialectic keenness, would have fixed a
limit to all competent investigation, if the striving after certainty were not
so ineradicably planted in the human breast. — Extract from author's preface.

Preliminary remark. I use in this memoir two different radical signs,
one which admits each root in \(\sqrt[n]{a}\), the other which always admits but one
of the values in the same formula. When this distinction is not given by
the usual symbols and I yet attach a particular weight to it, I remark that
here for the most part the reference is to such restricted values. In the few
places where this is not the case, it is known by the context.

§ 1.

I. \[ x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0 \]
is an equation of the 5th degree with the coefficients \(a, b, c, d, e\), and the
unknown quantity \(x\), whose value and form are to be determined.

After removing any existing coefficient of the term \(x^5\) by dividing the re-
main ing coefficients by it, I. is the most general form of the equation of the
5th degree: complete when \(a, b, c, d\) and \(e\) have significant values, incom-
plete when one or more of them \(= 0\).

If the roots of I. are taken as positive and rational the complete form
becomes

II. \[ x^5 - ax^4 + bx^3 - cx^2 + dx - e = 0, \]
which I retain as general.

By putting one or more of the coefficients \(a, b, c, d\) and \(e = 0\), several
varieties of this form present themselves, from which I particularly select:

III. (1) \[ x^5 + bx^3 - cx^2 + dx - e = 0, \]
the Capital, since it has properties which are of essential importance in
the solution of equations generally, and since it especially is the only form char-
acteristic of an exact group which is to be determined;

(2) \[ x^5 - e = 0, \]
which form, in restoration of an old notation, I name the Surd-solid.

I do not use any other than these forms.

§ 2.

The equation of the 5th degree has 5 unknown quantities: \(y, v, z, u, t\).
If any one, as \(y\), is known, the remaining 4 are found by an equation of
the 4th degree.
I denote the five roots of the equation \( x^5 - 1 = 0 \), or the 5 significations of the expression \( ^{5}\sqrt{1} \), by \( a(5)' \), \( a(5)'' \), \( a(5)''' \), \( a(5)'''' \), \( a(5)''''' \); I employ the notation \( ^{n}\sqrt{a} \) to denote that, in the formula in which it occurs, any value whatever of \( ^{n}\sqrt{a} \) is signified, but always the same one. Developing this, I find

\[
a(5)' = \frac{1}{4}[-1 + \sqrt{5} + \sqrt{(-5 + 2\sqrt{5})} + \sqrt{(-5 - 2\sqrt{5})}] \text{ or, more briefly,}
\]

\[
a(5)' = \frac{1}{4}[-1 + \sqrt{5} + \sqrt{(-5 + 2\sqrt{5})} + \sqrt{(-5 - 2\sqrt{5})}] = \frac{1}{4}[-1 - \sqrt{5} - p + q]
\]

\[
a(5)'' = \frac{1}{4}[-1 - \sqrt{5} - \sqrt{(-5 + 2\sqrt{5})} + \sqrt{(-5 - 2\sqrt{5})}] = \frac{1}{4}[-1 - \sqrt{5} - p - q]
\]

\[
a(5)''' = \frac{1}{4}[-1 + \sqrt{5} + \sqrt{(-5 + 2\sqrt{5})} - \sqrt{(-5 - 2\sqrt{5})}] = \frac{1}{4}[-1 + \sqrt{5} + p - q]
\]

\[
a(5)'''' = \frac{1}{4}[-1 + \sqrt{5} - \sqrt{(-5 + 2\sqrt{5})} - \sqrt{(-5 - 2\sqrt{5})}] = \frac{1}{4}[-1 + \sqrt{5} + p + q]
\]

\[
a(5)''''' = 1
\]

These expressions have the property that, if I arbitrarily denote any one of the imaginaries by \( a(5)' \), they are as \([a(5)'] : [a(5)'] : [a(5)'] : [a(5)'] : [a(5)'] \).

The proof is easy: If, e. g., \( (a')^n \) should be \( n \) and \( n \) no one of the remaining imaginary values of \( ^{n}\sqrt{1} \), then I would have \([ (a')^n ] = n \) = \( (a')^n (a')^n \)

\( = 1 \). Thus \( ^{n}\sqrt{1} = n \), and \( ^{n}\sqrt{1} \) has more than 5 values.

I designate the combination \( y.a(5)' \) briefly by \( y' \).

By considering this and the properties of \( a(5) \) already described, I have measurable simplified the operations: e. g., \( y'.v'' \) can now be written simply \( yv'' \), etc.

§ 3.

By regarding the equation

I.

\[ x^5 - ax^4 + bx^3 - cx^2 + dx - e = 0 \]

as the product of the differences of the general expression for its unknown quantity \( x \) and the special values \( y, v, z, u \) and \( t \), thus

II. \[ (x - y)(x - v)(x - z)(x - u)(x - t) = 0, \]

I have

III. \[ (y + v + z + u + t) = a, \]

IV. \[ (yv + yz + yu + yt + vz + vu + vt + xu + xt + ut) = b, \]

V. \[ (yvu + yvt + yzu + yst + yut + vzu + vst + vut + vul + vul) = c, \]

VI. \[ (yvut + yyst + yvst + yust + yvul) = d, \]

VII. \[ (yvust) = e, \]

from which I am in a position to form next the symmetrical functions of 3, 4, or 5 dimensions.
Having formed in this manner the values of \((y^6 + v^6 + z^6 + u^6 + \ell^6)\),
\((y^6 + v^6 + z^6 + u^6 + \ell^6)\),
\((y^4 + v^4 + z^4 + u^4 + \ell^4)\),
\((y^4 + v^4 + z^4 + u^4 + \ell^4)\),
the sums of the higher powers of \(y, v, z, u\) and \(t\) are obtained by putting

\[
\begin{align*}
\phi^6 &= ay^6 + by^4 + cy^2 + dy^0 - ey = 0 \\
v^6 &= ax^6 + bx^4 + cx^2 + dx^0 - ev = 0 \\
z^6 &= az^6 + bz^4 + cz^2 + dz^0 - ez = 0 \\
u^6 &= au^6 + bu^4 + cu^2 + du^0 - eu = 0 \\
\ell^6 &= a\ell^6 + b\ell^4 + c\ell^2 + d\ell^0 - e\ell = 0
\end{align*}
\]

\((y^6 + v^6 + z^6 + u^6 + \ell^6) = a(y^6 + v^6 + z^6 + u^6 + \ell^6) - b(y^4 + v^4 + z^4 + u^4 + \ell^4) + c(y^2 + v^2 + z^2 + u^2 + \ell^2) - d(y^0 + v^0 + z^0 + u^0 + \ell^0) + a\).

Thus I am able to represent every sum of equal powers of the unknown quantities in terms of the coefficients of \(I\), and also, by the help of the powers \((y + v + z + u + \ell)^n = a^n\), every symmetrical function of the same by means of such coefficients.

I denote a symmetrical function, e.g. that of \(b\), thus: \((yu \ldots \ldots . 10 t)\). In this expression \(yu\) signifies the form, and 10 the number of the terms.

§ 4.

The equation of the 5th degree permits the formation of 20 differences of its unknown quantities. The product of these differences is \(I. = (108b^3c^3 - 72b^3d^3 + 16b^3e^3 + 16b^2c^2d^2 - 900b^3d^3 + 825b^3d^2e + 56b^3e^2d + 128b^2c^2d^2 - 630b^2c^2e + 244b^2c^2d^2 - 3750b^3c^3 + 2000b^2c^2d^2 - 108c^3e - 27c^3d^2 + 2250c^2d^2e - 1600c^2d^2e + 256d^2e + 3125e^4)\).

I name ten of these differences in one direction, among which there are no two as + 1: — 1, e.g.,

\[
\begin{align*}
\text{II.} & \quad (y-v)(v-u)(z-t)(t-y)(y-z)(z-v)(v-t)(t-u)(u-y)(u-z); \\
\text{and the product of differences in the other direction is:} \\
\text{III.} & \quad (v-y)(u-v)(t-z)(y-t)(y-z)(z-y)(v-t)(t-u)(u-y)(z-u).
\end{align*}
\]

If expression \(I. = P\), both these products = \(\sqrt{P}\).

There are 12 such products of differences in one direction. This product represents a symmetrical function symmetrically divided, viz.:

\[
\text{IV.} \quad (y^2z^2u \ldots \ldots . 120 t). \quad \text{While one-half of this function is positive, the other is negative.}
\]

From the product of the differences in one direction for an equation of the 5th degree I am able, by assuming one unknown quantity as \(t = 0\), to

*Here, as in the equations of the 3d and 4th degrees the innermost square-root of the only irrational quantity occurring in the solution formula always has for its value the product of the differences of the respective unknown quantities, it is assumed that, in the complete representation of the formula for the roots of an equation of the 5th degree, the product of differences also occupies the same position.*
construct that of the biquadratic equation, and from this again, those of the cubic and quadratic equations. By doing this I recognize the law which underlies their formation and can now easily write, as by multiplication, the corresponding products of differences for an equation of the 6th degree, etc.

These products of differences in one direction can be again symmetrically analyzed.

§. 5.

I presuppose it known that I can replace a given equation of the 5th degree

I. \( x^5 - ax^4 + bx^3 - cx^2 + dx - e = 0 \) by its capital:

II. \( x_i^5 + b_i x_i^3 - c_i x_i^4 + d_i x_i - e_i = 0. \)

I put (one may arbitrarily assume it, on grounds, moreover, which relate to the characteristic of the group of equations whose capital is II.)

III. \( (-2a^2 + 5b) \)

\( (4a^3 - 15ab + 25c) \)

\( (-8ac + 35b + 20d) \)

\( (8a^2c - 3ab^2 - 50ad + 5bc + 250e) \)

= \( A \) as the first,

= \( B \) " second,

= \( C \) " third,

= \( D \) " fourth

constants of the group, and then for the capital there follows \( x_i = x + \frac{1}{4}a, \) &

IV. \( x_i^5 + \frac{Ax_i^3}{5} - \frac{Bx_i^2}{25} + \frac{(25C - 3A^2)x_i}{500} + \frac{(AB - 25D)}{6250} = 0. \)

Therefore I am authorized, henceforth, to operate freely with the capital as a basis, since I can obtain it for any equation, and can refer the result reached, with its application, to the given equation.

The use of the capital affords not only an aid to the calculation, by the elimination of \( a, \) but it is essentially necessary for a special purpose, viz., the partition of symmetrical functions.

If I multiply, by way of exercise, the term \( yvu, \) a part of \( (yvu + yvst + yuvt + vstu) = d, \) by \( (y + v + s + u + t) = a = 0, \) there results

\( y^2vu + v^2yvu + s^2yvu + u^2yvu + vstu = 0. \)

Thus:

\( (y^2vu + v^2yvu + s^2yvu + u^2yvu)= - e. \)

The symmetrical function \( (y^2vu \ldots 20t) \) thus admits of division into 5 parts of equal value \( -e. \)

§. 6.

An equation of the 5th degree is solvable if the constants

\( A \) or \( B \) and \( D \) or

\( A \) and \( B \) and \( C \)

each = 0, or if \( AB = 25D, \)
because then the capital will be replaced by an equation of the 4th degree or the surd-solid. In this manner we can transform every equation into one in which one of the first three constants (but not \( D \)) = 0; but since a general solution of an equation of the 5th degree is approached no nearer by this path, I pursue it no further.*

§ 7.

Let there be given: \( a^5 + bx^3 - cx^2 + dx - e = 0 \).

This equation is solved when I have represented its unknown quantity \( y \) by a function of \( b, c, d \) and \( e \).

And this function can contain only one irrational quantity under an outer radical of the 5th degree, since otherwise it would have and therefore \( y \), more than a quintuple signification.

Further, upon the ground just stated, the expression for the unknown \( y \) can contain no quantity with an outer radical of higher degree than the 5th.

Whatever is the form of the dignant\( ^\dagger \) of the expression \( \sqrt[5]{F} \), which may represent this radical of the 5th degree, it may remain provisionally uncertain; but if it contains radical quantities they must occur with exactly determinate values and sustain to one another a common relation in all the formulas, since otherwise again more than five values would be given for \( y \); this value must be relatively rational and susceptible of only one signification.

A similar condition will hold in regard to the coefficients occurring with \( \sqrt[5]{F} \) or the powers of this expression.

It is evident, on the other hand, that that one radical quantity can occur in the 1st, 2d, 3d and 4th powers, whereby the chief requisite of the expression, that it yield only five values, is not violated.

By considering this I am able to separate the expression for \( y \) into groups, of which one embraces the sum of the terms without the factor \( \sqrt[5]{F} \), while four others have, each one, factors which are respectively the 1st, 2d, 3d and 4th powers of \( \sqrt[5]{F} \). Thus:

\[
\Pi. \quad y = A + B\sqrt[5]{F} + C(\sqrt[5]{F})^2 + D(\sqrt[5]{F})^3 + E(\sqrt[5]{F})^4,
\]

or with briefer notation:

---

*With the equation of the 3d degree the corresponding process leads to a new mode of solution; and with that of the 4th degree an analogous procedure leads to a solution by a quadratic auxiliary equation.

\( ^\dagger \)As this term is always used with a special signification, I simply transfer it from the original.
\[ y = A + Bf + Cf^2 + Df^3 + Ef^4 \]
\[ v = A + Bf' + Cf'' + D(f')^3 + Ef(f')^4 \]
\[ z = A + Bf'' + Cf''' + D(f'')^3 + Ef(f'')^4 \]
\[ u = A + Bf''' + Cf'''' + D(f''')^3 + Ef(f''')^4 \]
\[ t = A + Bf'''' + Cf''''' + D(f''''')^3 + Ef(f''''')^4 \]

but since \((f')^2 = f''\), &c., I write the above thus:

III.
\[ y = A + Bf + Cf^2 + Df^3 + Ef^3 \]
\[ v = A + Bf' + Cf'' + Df''' + Ef'''' \]
\[ z = A + Bf'' + Cf''' + Df'''' + Ef''''' \]
\[ u = A + Bf''' + Cf'''' + Df''''' + Ef'''''' \]
\[ t = A + Bf'''' + Cf''''' + Df''''' + Ef'''''' \]

Sum: \((y + v + z + u + t) = 5A = a\), or in this case \((\text{capital}) = 0\).

By now removing \(A = 0\) and multiplying \(y, v, z, u\) and \(t\) respectively by \(a(5)', a(5)'', \text{etc.}, \) so that \(Ef^3\) always enters without the imaginary \(5\sqrt{1}/1\), I have:

IV.
\[ y = Bf + Cf^2 + Df^3 + Ef^3 \]
\[ v' = Bf' + Cf'' + Df''' + Ef'''' \]
\[ z'' = Bf'' + Cf''' + Df'''' + Ef''''' \]
\[ u''' = Bf''' + Cf'''' + Df''''' + Ef'''''' \]
\[ t''' = Bf'''' + Cf''''' + Df''''' + Ef'''''' \]

\((y + v' + z'' + u''' + t''') = 5Ef^3;\) similarly I find
\((y + v'' + z''' + u' + t''') = 5Df^3\)
\((y + v''' + z' + u'''' + t''') = 5Cf^2\)
\((y + v'''' + z'' + u''' + t') = 5Bf,\)

whereby it is proved that the roots of an equation of the 5th degree must always be represented as an aggregate of the sums:

V. \[ y = \left\{\frac{(y + v + z + u + t) + (y + v' + z'' + u''' + t''') + (y + v'' + z''' + u'''' + t'' ) + (y + v''' + z''' + u'''' + t') + (y + v'''' + z'' + u''' + t'''}{5}\right\} \]

These sums I call solution-sums. The mutual fitness of any five of these is given by their origin. One of them, the basis of the group, is so chosen that in it one of the unknown quantities is combined with \((5\sqrt{1})',\) the second with \((5\sqrt{1})'',\) the third with \((5\sqrt{1})''',\) the fourth with \((5\sqrt{1})''''\), and finally the 5th with \((5\sqrt{1})''''' = 1.\) The four remaining solution-sums belonging to expression V. are now constructed on this basis by combining the 2d, 3d, 4th and 5th powers of the relative \(5\sqrt{1}\) with the same unknown quantities.\(^*\)

\(^*\)By the 24 possible permutations of \(v, z, u\) and \(t,\) 23 other groups of such aggregates can
In this case I employ \( Bf = (y + v'' + z'' + w'' + t') \) as the basis. I assume the factor \( B \) of the same \( = 1 \), whereby nothing in the signification of the formula is changed (only the meaning of the individual values \( f, C, D, E \) while the result remains the same, and I write

VI.

\[
\begin{align*}
y &= \frac{i}{4} \left[ f + C f^3 + D f^3 + E f^3 \right] \\
v &= \frac{i}{4} \left[ f' + C f' + D f' + E f' \right] \\
z &= \frac{i}{4} \left[ f'' + C f'' + D f'' + E f'' \right] \\
u &= \frac{i}{4} \left[ f''' + C f''' + D f''' + E f''' \right] \\
t &= \frac{i}{4} \left[ f'' + C f'' + D f'' + E f'' \right]
\end{align*}
\]

If from this I deduce the symmetrical functions for \( b, c, d \) and \( e \), there results \( f^5 = F \):

VII. 

(1) \( b = -\frac{i}{4} [E + CD] F \)

(2) \( c = \frac{1}{16} \left[ (C^3 + D^3) F^3 + (C^3 + D) F \right] \)

(3) \( d = \frac{1}{16} \left[ -DE^3 F^3 + (-C^6 + C^2 D^2 - CDE - D^3 + E^2) F^2 - CF \right] \)

(4) \( e = \frac{1}{8} \left[ (E^6 F^4 + D^6 F^4 + C^6 F^4 + F^6) + 5(C^2 D^2 E^2 - C^2 D^2 E^2) \right] \)

Equations VII. (3) and (4) are yet to be abbreviated. From (1) and (2):

(5) \( 25b^3 = (C^2 D^2 F^2 + E^2 F^2 + 2CD EF) \)

(6) \( 125bc = (C^4 D^2 E^2 F^3 + C^2 D^2 E^2 F^3 + C^6 F^3 + D^2 E^3 F^3 + C^2 DF^3 \)

(7) \( 125d - 25b^2 = -(C^3 E^5 + 3C D E + D^3) F^3 - CF \)

(8) \( -125bc + 3125c = (E^6 F^4 + D^6 F^4 + C^6 F^4 + F^4) + 10(C^2 D^2 E^2 \)

\( + D^2 E^4) F^3 + 10(C^2 D E + C D^2 E^2) F^3 \)

(To be continued.)

be formed. It is the same whichever one of these is employed in the representation of the formula for \( y \), as in the equation of the third degree

\[
3 \sqrt[3]{\frac{B + \sqrt{B^2 - 4A^3}}{2}} + 3 \sqrt[3]{\frac{B - \sqrt{B^2 - 4A^3}}{2}} = A
\]

represents the unknown quantity \( y \) as well as

\[
3 \sqrt[3]{\frac{B - \sqrt{B^2 - 4A^3}}{2}} + 3 \sqrt[3]{\frac{B + \sqrt{B^2 - 4A^3}}{2}} = A
\]

Both are aggregates of solution-sums: in this case certainly the same, since the 3d degree has to employ for the representation of \( y \) only the three forms \( (y + v + z) \), \( (y + w + z) \) and \( (y + w + z) \).

If now in V. \( y \) is admitted to the possible permutations, 120 solution-sums arise, whose application to the representation of the roots of the 5th degree is given in § 4, § 7 and 8.

*We must now, for example, regard \( y \) as written:

\[
\]
A NEW METHOD OF SOLVING NUMERICAL EQUATIONS.

BY DR. H. EGGEN, MILWAUKEE, WISCONSIN.

Among the methods for solving numerical equations, there are such as teach, to find from one or more initial approximate values a more approximate one of the root required. Such methods are, e. g., the method of Newton and the rule of false position. These methods become in certain cases impracticable because of slow convergence. The Newtonion method, e. g., converges very slowly toward a multiple root. For in the case of a simple root the convergence is of quadratic order, while in the case of a multiple root the acceleration is only linear. That is to say: in case of a simple root every correction is proportional to the square of the preceding correction; but in case of a multiple root every correction is proportional to the first power of the preceding correction.

It is true, the multiple roots can always be separated from the given equation, but the troublesome case of nearly equal roots remains. Besides this, the labor of separating multiple roots is not always small, and it is therefore desirable to avoid it, if possible.

The object of this essay is, to communicate a method of calculation which is independent of the relations of roots to one another, to wit, a method which converges with the same rapidity whether the roots be simple or multiple: The acceleration towards the roots is in every case of quadratic order. This algorithm is the following:

Theorem. Let \( f(x) = 0 \) be a numerical equation of the \( n \)th degree, \( x_0 \) an arbitrary approximate value of one of its roots, \( f'(x) \) and \( f''(x) \) the first and second differential quotients, then the next corrected value \( x_1 \) is found to be

\[
x_1 = x_0 - \frac{f(x_0) f'(x_0)}{[f'(x_0)]^2 - f''(x_0)}
\]

or briefly

\[
x_1 = x_0 - \frac{f f'}{f'^2 - f''} = F(x_0).
\]

The next corrected value \( x_2 \) would be the same function of \( x_1 \) as \( x_1 \) is of \( x_2 \), etc.

Now let \( x \) be one of the roots of the given equation \( f(x) = 0 \). In this case we would derive from equation (2) \( x_1 = x \); for the correction would vanish, as \( f(x) \) is a factor of it. Further, let us suppose \( x_0 \) to be an approximate value of this root, so that

\[
x_0 = x + \epsilon,
\]
then the corrected value \( x_1 \) would no more be equal to \( x_0 \), but we would have, according to (2):

\[
x_1 = F(x + e) = F(x) + eF'(x) + \frac{e^2}{2}F''(x) + \frac{e^3}{6}F'''(x) + \ldots
\]

and \( x_1 \) would differ from the root by some quantity \( e_1 \), therefore

\[
x_1 = x + e_1 = F(x) + eF'(x) + \frac{e^2}{2}F''(x) + \ldots
\]

Now \( x = F(x) \), consequently we get an expression for \( e_1 \),

\[
e_1 = eF'(x) + \frac{e^2}{2}F''(x) + \frac{e^3}{6}F'''(x) + \ldots
\]

The function \( F'(x) \), the first differential quotient of \( F(x) \) possesses the property (and this is essential for the establishment of our method) that, the first differential quotient, \( F'(x) \), vanishes for \( f(x) = 0 \), even in the case when the root \( x \) is a multiple root of any order. Therefore the error \( e_1 \) of the corrected value of the root \( x_1 \) is

\[
e_1 = \frac{e^2}{2}F''(x) + \frac{e^3}{6}F'''(x) + \ldots
\]

to wit, the new error \( e_1 \) is proportional to the square of the preceding error, whether the root in question is simple or multiple; that is, the acceleration of our method is always of second order, whereas the method of Newton,

\[
x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}
\]
in case of a multiple root furnishes a convergence of linear order.

If it should happen that \( F'''(x) \) vanishes for \( f(x) = 0 \), the acceleration would be of the third order.

This algorithm is only one particular case of an infinite number of methods of every possible rate of convergence. But the amount of work connected with those algorithms of greater convergence is not fully balanced by the greater rapidity of convergence. For example, an algorithm of cubic acceleration, for simple or multiple roots, is the following:

\[
x_1 = x_0 - \frac{f f''' - f' f''}{(f')^3 - \frac{1}{2} f f'' f' + \frac{1}{6} f'^3 f''}
\]

where \( f, f', f'' \) are functions of \( x_0, f' \) the first, and \( f'' \) the second differential quotients of the given equation \( f(x) = 0 \).

As it would require considerable space to prove the above statements, I suppress the demonstration here, but I may be permitted to add one example:

**Ex.** Given the equation \((x - 2)^8 (x - 4) = 0\) with the multiple root 2. Taking for initial value \( x_0 = 3 \), we find successively

\[
x_1 = 2.5, \quad x_2 = 2.07, \quad x_3 = 2.0009, \quad x_4 = 2.0000001, \text{ etc.}
\]

**Observation.** Always that root is found the modulus of which is the nearest to the modulus of the arbitrary initial trial value. If in the foregoing example we would commence with \( x_0 = 5 \), we would approach the root 4.
TRISECTION OF AN ANGLE.

BY DR. WILLIAM HILLHOUSE, NEWHAVEN, CONN.

Draw a straight line $OD$, and on it erect a perpendicular $HM$. Take any point $F$, in $HM$ and from $F$ with twice the distance $FH$, as radius, describe an arc cutting the line $OD$ in $O$. With $O$ as center and radius $FH$, (or half $OF$,) describe the arc $ABC$. Again, from the point $O$, lay off, on the line $OH$, three times $OA$, to the point $D$, and with $H$ as center and radius $HD$, describe the arc $DEL$.

To trisect any angle from 0 to $90^\circ$, say $AOC$, draw $OC$, making $AOC$ = the given angle; through $C$ draw $CE$ parallel to $OD$ and draw the radius $HI$ parallel to $OC$. Draw $IP$ perpendicular to $CE$ and join $OP$, cutting the arc $AC$ in $B$, then will the angle $AOB$ equal one-third of the angle $AOC$.

[The foregoing construction is somewhat remarkable in that it gives accurate results at the two limits of an arc of $90^\circ$; that is, the trisection is exact for an arc of $90^\circ$ and also for an infinitely small arc, and is a close approximation for all intervening arcs; so close, indeed, that a few trial constructions would be likely to induce a casual observer to believe that the trisection is exact.—That it is not, may be proved as follows:]

Assume that the trisection, as represented in the figure, is exact. Then, from a consideration of the triangles $OHS$ and $OHR$, we can easily prove that $OR = 2HS$.

From the similar triangles $OHS$ and $PTS$ we find

$$OR = \frac{s \sqrt{[(rv + s)^2 + 2e(rv + s) + 1]}}{rv + v + s}$$

and

$$HS = \frac{s}{rv + s}$$

where $v =$ cosine of the angle to be trisected, $s = OH = \sqrt{3}$ and $r = TI = 2 - \sqrt{3}$. If the construction is exact we shall therefore have
\[
\frac{s \sqrt{[(rv + s)^2 + 2c(rv + s) + 1]} = \frac{2s}{rv + s}}{rv + v + s}
\]

Whence \(\sqrt{[(rv + s)^2 + 2c(rv + s)^2 + (rv + s)^2]} = 2(rv + v + s)\).

Because \(v\) cannot be eliminated from this equation therefore the trisection cannot, in general, be exact; otherwise we would have an equation of the 4th degree with an infinite number of roots.

Moreover, as the above equation is verified by substituting for \(v\) its value at either limit, and as it has two negative roots, it follows that the foregoing construction will not trisect any finite angle except one of 90°.—Ed.]

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**SOLUTION OF A PROBLEM IN PROBABILITIES.**

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BY THE EDITOR.

If four bricks are placed on each other at random, with their longest axes horizontal and in the same vertical plane, determine the probability that the pile will stand.*

Let the bricks, in the order in which they are placed on the pile, be represented respectively by (1), (2), (3), (4).

Denote the length of a brick by \(a\), and the horizontal distance between the centers of (1) and (2), (2) and (3), (3) and (4), respectively, by \(x, y, z\).

Then, if \(x\) be not greater than \(\frac{1}{4}a\), and if \(y\) is estimated in the same direction from the center of (1) as \(x\) is, and is taken not greater than \(\frac{1}{4}a\), the center of (4) may take any position on (3) and the pile will stand. This 1st case therefore gives, for the number of stable positions of the pile,

\[\frac{1}{4}a \times \frac{1}{4}a \times \frac{1}{4}a = u_1 = \frac{1}{4}\alpha^2.\]

Also, while \(x\) is restricted to \(\frac{1}{4}a\), \(y\) may take any value between \(\frac{1}{4}a\) and \(\frac{1}{4}a\), but the range of the center of (4), for stable positions of the pile, will now be \(\frac{1}{2}(3\alpha - 4y)\); hence the number of stable positions in this case will be

\[\frac{1}{4}a \int_{\frac{1}{4}a}^{\frac{1}{4}a} \frac{1}{2}(3\alpha - 4y)dy = \frac{1}{4}a \left[ \frac{1}{8}ay - y^2 \right]_{\frac{1}{4}a}^{\frac{1}{4}a} = u_2 = \frac{1}{12}\alpha^2.\]

In like manner we may find that for values of \(x\) between \(\frac{1}{4}a\) and \(\frac{1}{4}a\), \(y\) may have any value between 0 and \(\frac{1}{4}(\alpha - 3\alpha)\), and \(z\) any value from \(-\frac{1}{4}a\) to \(\frac{1}{4}a\); hence the number of stable positions will be

*This problem was, I believe, first proposed by Artemas Martin, in the Schoolday Visitor for Dec. 1871. No solution was published in that periodical, nor elsewhere so far as I know, though two other solutions of the question have been made, independently; one by E. B. Seitz and one by Henry Heaton.
\[ a \int_0^{\frac{1}{3}a} \int_0^{\frac{1}{3}(a-3x)} dx \, dy = \frac{1}{4}a \left[ ax - \frac{3}{4}x^2 \right]_{\frac{1}{3}a}^{\frac{1}{3}a} = u_4 = \frac{1}{4}a^3. \]

Also, for the same values of \( x \), if \( y \) be taken between the limits \( \frac{1}{3}(a - 3x) \) and \( \frac{1}{3}a \), the pile will stand for all positions of the center of (4) included in the expression \( 2a - 3x - 2y \). Therefore, in this case we have

\[ \int_0^{\frac{1}{3}a} \int_0^{\frac{1}{3}(a-3x)} (2a - 3x - 2y) dx \, dy = \left[ \frac{1}{4}ax^2 - \frac{3}{4}x^3 \right]_{\frac{1}{3}a}^{\frac{1}{3}a} = u_4 = \frac{1}{4}a^3. \]

If for values of \( x \) between \( \frac{1}{3}a \) and \( \frac{1}{3}a \), \( y \) be taken between the limits 0 and \( \frac{1}{3}(a - 3x) \) the range of the center of (4) for stable positions of the pile will again be \( a \), and we shall have

\[ a \int_0^{\frac{1}{3}a} \int_0^{\frac{1}{3}(a-3x)} dx \, dy = \frac{1}{4}a \left[ ax - \frac{3}{4}x^2 \right]_{\frac{1}{3}a}^{\frac{1}{3}a} = u_5 = \frac{1}{14}a^3. \]

Also if for the same values of \( x \) as in the last case \( y \) take any value between \( \frac{1}{3}(a - 3x) \) and \( a - 2x \), in all case of stability of the pile, the center of (4) will again be confined to the line \( 2a - 3x - 2y \), and we shall have

\[ \int_0^{\frac{1}{3}a} \int_0^{\frac{1}{3}(a-3x)} (2a - 3x - 2y) dx \, dy = \left[ \frac{1}{4}ax^2 - \frac{3}{4}x^3 \right]_{\frac{1}{3}a}^{\frac{1}{3}a} = u_5 = \frac{1}{14}a^3. \]

If we allow \( x \) to take any value between \( \frac{1}{3}a \) and \( \frac{1}{3}a \), \( y \) may have any value between 0 and \( 2a - x \), and the pile will stand for all positions of the center of (4) as in the last case, and we shall have

\[ a \times \frac{1}{3}a \times a = u_5 = \frac{1}{4}a^3. \]

If while \( x \) is estimated in either direction from the center of (1) \( y \) is taken in the opposite direction, the stable positions of the pile will be as follows: — If \( x \) be taken between 0 and \( \frac{1}{3}a \), and \( y \), between 0 and \( \frac{1}{3}a \), the center of (4) may take any position on (3). Hence we have

\[ \frac{1}{3}a \times \frac{1}{3}a \times a = u_6 = \frac{1}{2}a^3. \]

If \( x \) be taken between 0 and \( \frac{1}{3}a \), and \( y \), between \( \frac{1}{3}a \) and \( \frac{1}{3}a \) the center of (4) must be confined to the line \( \frac{1}{3}(3a - 2y) \), and we get

\[ \frac{1}{3}a \int_0^{\frac{1}{3}a} \frac{1}{3}(3a - 4y) dy = \frac{1}{3}a \left[ \frac{3}{2}ay - y^2 \right]_{\frac{1}{3}a}^{\frac{1}{3}a} = u_6 = \frac{1}{2}a^3. \]

There remains only the case in which \( x \) varies from \( \frac{1}{3}a \) to \( \frac{1}{3}a \), and \( y \), from 0 to \( \frac{1}{3}a \); and in this case, if, while \( x \) takes any value from \( \frac{1}{3}a \) to \( \frac{1}{3}a \), \( y \) vary within the limits 0 and \( \frac{3}{2}x - \frac{1}{3}a \), the center of (4) will be confined to the line \( 2a - 3x + 2y \), and we shall have

\[ \int_0^{\frac{1}{3}a} \int_0^{\frac{1}{3}(3a-3x)} (2a - 3x + 2y) dx \, dy = \left[ -\frac{3}{4}x^2 + \frac{3}{4}ax^2 - ax^2 \right]_{\frac{1}{3}a}^{\frac{1}{3}a} = u_7 = \frac{1}{4}a^3. \]

Finally, with the same limits for \( x \) as in the last case, if \( y \) vary from \( \frac{3}{2}x - \frac{1}{3}a \) to \( \frac{1}{3}a \) the center of (4) may take any position on (3), and we shall have
\[ a \int_{1/a}^{1/a} \int_{1/(3a-x)}^{1/a} dx \, dy = \frac{1}{6} ax^2 = u_{11} = \frac{1}{8} a^2. \]

As the whole number of different trials that can be made will be represented by \(8a^3\), if \(P\) represent the required probability, because the stable positions above calculated are just half the whole number of stable positions, (\(x\) having been taken only from the center of (1) to one extremity,) we shall have

\[ P = \frac{u_1 + u_2 + \ldots + u_{11}}{4a^3} = \frac{209}{2304}. \]

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**SOLUTION OF TWO PROBLEMS IN SUMMATION OF SERIES.**

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**BY PROF. D. TROWBRIDGE, WATERBURGH, N. Y.**

1. **Required** the sum of the series

\[ 1 + \frac{1}{1 + r} + \frac{1}{1 + 2r} + \frac{1}{1 + 3r} + \ldots + \frac{1}{1 + (n-1)r} = \sum_{r}^{n}. \ldots (1) \]

**Solution.**—First find the value of the definite integral

\[ \frac{1}{r} \int_{-1}^{0} y^r dy(1+y)^r = \left[ y^r(1+y)^r \right]_{-1}^{0} - \frac{P^{r-1}r^r(1+y)^{(r+1)r-1}}{1 + r} \]

\[ + \frac{P(r-1)y^{r-2}r^r(1+y)^{(r+2)r-1}}{(1+r)(1+2r)} + (-1)^r \cdot \frac{P[r - 1] \ldots 2.1.r}(1+r)(1+2r) \ldots (1+pr) \]

\[ = (-1)^r \cdot \frac{P(r-1) \ldots 2.1.r}{(1+r)(1+2r) \ldots (1+pr)} = V_r. \ldots (2) \]

Make \(p = 0, 1, 2, 3, \ldots \), successively, and we shall have

\[ V_0 = 1, \quad V_1 = -\frac{r}{1 + r}, \quad V_2 = \frac{2r^3}{(1+r)(1+2r)}, \quad V_3 = -\frac{3r^3}{(1+r)(1+2r)(1+3r)}, \ldots \ldots (3) \]

Now take the series

\[ 1 + x + x^2 + \ldots + x^{(n-1)r} = \frac{x^{n-1} - 1}{x - 1}, \quad \text{then} \]

\[ \sum_{r}^{n} = \int_{0}^{1} dx(1 + x + x^2 + \ldots) = \int_{0}^{1} dx. \frac{x^{n-1} - 1}{x - 1} \]

Put \(x = 1 + y, \quad x = x/(1 + y), \quad dx = (1 + r) \cdot (1 + y) dy, \) and

\[ \frac{x^{n-1} - 1}{y} = n + \frac{n(n-1)}{1.2.3} + \ldots \]

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\[ \Sigma_0^{(r)} = \int_0^1 \frac{n(n-1)^r}{1.2} \left[ (1+y)^{\frac{1}{r} - 1} + \frac{n(n-1)y}{1.2} (1+y)^{\frac{1}{r} - 1} + \cdots \right] dy. \] 

\[ \Sigma_1^{(r)} = n \frac{n(n-1)}{1.2} + \frac{n(n-1)(n-2)r^3}{1.23.1} - \frac{n(n-1)(n-2)(n-3)r^3}{1.23.4.1} + \cdots \] 

\[ \Sigma_1^{(r)} = n \frac{n(n-1)}{1.2.2} + \frac{n(n-1)(n-2)}{1.2.3.2} - \frac{n(n-1)(n-2)(n-3)}{1.2.3.4} + \cdots \] 

(4) 

From (4) we easily find 

\[ \Sigma_{r+1}^{(r)} - \Sigma_r^{(r)} = \frac{1}{1+nr} = 1 - \frac{nr}{1+r} + \frac{n(n-1)r^3}{1.2(1+r)(1+2r)} - \frac{n(n-1)(n-2)r^3}{1.23.1} + \cdots \] 

\[ \frac{1}{1-nr} = 1 + \frac{nr}{1+r} + \frac{n(n-1)r^3}{1.2(1+r)(1+2r)} + \frac{n(n-1)(n-2)r^3}{1.23.1} + \cdots \] 

\[ \frac{1}{1-n^2r} = 1 + \frac{nr}{1+r} + \frac{n(n-1)r^3}{1.2(1+r)(1+2r)} + \frac{n(n-1)(n-2)r^3}{1.23.1} + \cdots \] 

2. Find the sum of the series 

\[ S_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2} \] 

(6) 

Since \( 1 + x + x^2 + x^3 + \cdots + x^{r-1} = x^{r-1} / (x-1) \) we evidently have 

\[ S_n = \int_0^x \int_0^x \frac{x^r - 1}{x-1} dx \] 

(7) 

Put \( x-1 = y, \ dx = dy \), then 

\[ \frac{x^r - 1}{x-1} = \frac{(1+y)^r - 1}{y} = n + \frac{n(n-1)y}{1.2} + \frac{n(n-1)(n-2)y^2}{1.2.3} + \cdots \] 

\[ \int_0^x \frac{x^r - 1}{x-1} \ dx = \int_0^y \left[ n + \frac{n(n-1)y}{1.2} + \cdots \right] dy + \frac{n(n-1)(n-2)y^2}{1.2.3} + \cdots \] 

\[ = n(y+1) + \frac{n(n-1)y^2}{1.2} + \frac{n(n-1)(n-2)y^3}{1.2.3} + \cdots \] 

Therefore, since \( dx / x = dy / (1+y) \) 

\[ S_n = \int_0^\infty dy \left[ n + \frac{n(n-1)(y-1)}{1.2.2} + \frac{n(n-1)(n-2)(y^2 - y + 1)}{1.2.3.3} + \cdots \right] = n \] 

\[ = \frac{n(n-1)(1+1)}{1.2.2} + \frac{n(n-1)(n-2)(1+1)}{1.2.2} + \frac{n(n-1)(n-2)(n-3)(1+1)}{1.2.3.4} + \cdots \] 

\[ \therefore S_n = n - \frac{n(n-1)\Sigma_0^{(1)}}{1.2.2} + \frac{n(n-1)(n-2)\Sigma_1^{(1)}}{1.2.3.3} - \frac{n(n-1)(n-2)(n-3)\Sigma_3^{(1)}}{1.2.3.4.4} + \cdots \] 

(8)
SOLUTION OF PROBLEM 118.

BY PROF. JOHNSON.

Denoting the pairs of opposite sides and diagonals by $a, a'; b, b';$ and $c, c'$; and the angles between $b$ and $c$, $c$ and $a$, and $a$ and $b$ respectively by $\alpha, \beta$ and $\gamma$ we have $\alpha + \beta = \gamma$, from which we derive the relation between cosines

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma = 1. \ldots \ldots (1)$$

By the triangles

$$\cos\alpha = \frac{b^2 + c^2 - a'^2}{2bc},$$
$$\cos\beta = \frac{c^2 + a^2 - b'^2}{2ac},$$
$$\cos\gamma = \frac{a^2 + b^2 - c'^2}{2ab}.$$

Substituting in (1)

$$b^4(a^2 + a'^2 - b'^2) + a^2(b^2 + c^2 - a'^2) + c^2(a^2 + b^2 - c'^2) - (a^2 + a'^2 - b'^2)(b^2 + c^2 - a'^2) \times (a^2 + b^2 - c'^2) - 4a^2b^2c^2 = 0,$$

which reduces to

$$a^2a'^2(a^2 + a'^2 - b'^2 - b'^2 - c'^2 - c'^2) + b^2b'^2(b^2 + b'^2 - c'^2 - c'^2 - a'^2 - a'^2) + c^2c'^2(c^2 + c'^2 - a^2 - a'^2 - b^2 - b'^2) + a^2b'^2c'^2 + b^2c^2a'^2 + c^2a'b'^2 + a^2b'^2c'^2 = 0. \ldots \ldots (2)$$

In the special case where $a = a'$ and $b = b'$ (2) reduces to

$$2a^2(a^2 - b^2)^2 - 2b^2(a^2 - b^2)(a^2 + b^4 + c^2a'^2)(c^2 + c'^2) - 2(a^2 + b^4)c^2c'^2 = 0$$

or

$$[a^2c'^2 - (a^2 - b^2)^2][c^2 + c'^2 - 2(a^2 + b^4)] = 0.$$

Hence, either

$$c^2 + c'^2 = 2(a^2 + b^4), \ldots \ldots \ldots \ldots (3)$$

or

$$c'^2 = a^2 - b^4. \ldots \ldots \ldots \ldots (4)$$

(3) holds when $a, a', b$ and $b'$ form a parallelogram and (4) holds when these lines form the crossed quadrilateral known as the contra-parallelogram, $c$ and $c'$ are parallel and have a constant product, as first observed by Mr. Henry Hart, who employed a linkage of this form as a reciprocator. (See Analyst, Vol. III, p. 46 or Vol. II, p. 44.)

[As Prof. Johnson claims that the solution of this question, as published at page 126, is not sufficiently general to meet the requirements of the question as proposed, we insert the above and add the remark, that the conclusion of Prof. Scheffer's solution, in which he had discussed the crossed quadrilateral, was omitted in the printed solution to avoid the necessity of introducing another diagram.]
SOLUTION OF PROBLEMS IN NUMBER FOUR.

Solutions of problems in No. 4 have been received as follows:
From R. J. Adcock, 120 and 123; Marcus Baker, 119, 122 and 123;
Dr. H. Eggars, 122; Edgar Frisby, 121; Henry Heaton, 119, 120, 121*,
122, 123 and 124; Prof. H. T. J. Ludwick, 121*; L. Regan, 119 and
122; E. B. Seitz, 119, 122, 123 and 124.

119. "If from one of the extremities, B, of the diameter, AB, of a given
circle, any chord, BD, be drawn, and from the other extremity, A, and
the centre C, two lines, AE, CE, be drawn to any point, E, in the circum-
ference, cutting said, chord in the points F and G; then, GF : GD ::
(BF)2 : (BA)2. Required the demonstraton."

DEMONSTRATION BY E. B. SEITZ, GREENVILLE, OHIO.

Draw GH parallel to BA; join DE and DGF and HGF give DF : HF :: EF : GF;
or DF : EF :: HF : GF; hence the triangles DFH and EFG are similar; therefore
∠HDG = ∠GEH = ∠A, and consequently the triangles ABF and DGH are similar.

The similar triangles ABF, HGF and DGH give GF : GH :: BF : BA, and GH : GD :: BF : BA. Multiplying
these proportions together, we have GF : GD :: BF2 : BA2.

120. [As the solutions of this question, that have been submitted, are
only forms from which the numerical value of x and y can be found by ap-
proximation, and as the answer which was sent with the question by Dr.
Oliver, the proposer, cannot be verified by assigning particular values to x
and y, and is therefore erroneous, it is not likely that a general solution of
the question can be obtained.]

121 "In a spherical triangle are given the sum of each angle and the
side opposite, to solve the triangle."

SOLUTION BY EDGAR FRISBY, ESQ., NAVAL OBSERVATORY, WASH., D.C.

We have
\[
\begin{align*}
\frac{\sin A - \sin a}{\sin A + \sin a} &= \frac{\sin B - \sin b}{\sin B + \sin b} = \frac{\sin C - \sin c}{\sin C + \sin c} = \tan \frac{1}{2}(A-a) &\text{&c.} = x \text{say.}
\end{align*}
\]
(2) \[
\tan \frac{1}{4}(A-a) = k_x, \tan \frac{1}{4}(B-b) = m_x, \tan \frac{1}{4}(C-c) = n_x \text{ if } \tan \frac{1}{4}(A+b) = l \text{ and }
\]
\[
\sin A = \left[\tan \frac{1}{4}(A+a) + \tan \frac{1}{4}(A-a)\right] \cos \frac{1}{4}(A+a) \cos \frac{1}{4}(A-a)
\]
\[
\cos A = \sqrt{\left[1 + \frac{1}{4}x^2\right]} \quad \sin a = \sqrt{\left[1 + \frac{1}{4}x^2\right]}
\]
\[
\cos a = \frac{1}{\sqrt{[1 + \frac{1}{4}x^2]}}
\]
and similar expressions for \(B, b, C, c\); substituting these values in Cagnoli's equation

(4) \[
\sin B \sin C - \sin b \sin c = \cos B \cos C \cos a + \cos b \cos c \cos A
\]
we have

(5) \[
\frac{mn\left[(1 + x)^2 - (1 - x)^2\right]}{\sqrt{\left[1 + m^2\right] \left[1 + n^2\right] \left[1 + x^2\right]}} = \frac{(1 - m^2 x)(1 - n^2 x)(1 + x^2) - (1 + m^2 x)(1 + n^2 x)(1 - x^2)}{\sqrt{\left[1 + m^2\right] \left[1 + n^2\right] \left[1 + x^2\right]}}
\]
or

(6) \[
2mn\sqrt{\left[1 + m^2\right] \left[1 + n^2\right] \left[1 + x^2\right]} = 1 + (m^2 n^2 - l^2 n^2 - l^2 m^2)x^2,
\]
which, on clearing of radicals, becomes

\[
1 - 2(l^2 m^2 + m^2 n^2 + n^2 l^2 + 2l^2 m^2 n^2)x^2 + (l^4 m^2 + m^4 n^2 + n^4 l^2 - 2l^2 m^2 n^2 - 2l^2 m^2 n^2 - 4l^2 m^2 n^2)x^2 = 0;
\]

dividing by \(x^2\) and solving we have

(7) \[
x^{-2} = l^2 m^2 + m^2 n^2 + n^2 l^2 + 2l^2 m^2 n^2 + 2l m n \sqrt{\left[1 + m^2\right] \left[1 + n^2\right] \left[1 + x^2\right]},
\]
or

(8) \[
1 - x^{-2} = (1 + l^2)(1 + m^2)(1 + n^2) - l^2(1 + m^2)(1 + n^2) - m^2(1 + n^2)(1 + l^2)
\]

\[
- n^2(1 + l^2)(1 + m^2) + 2l m n \sqrt{\left[1 + m^2\right] \left[1 + n^2\right] \left[1 + x^2\right]}
\]

\[
= (1 + l^2)(1 + m^2)(1 + n^2)
\]

\[
\times \left(1 - \frac{l^2}{1 + l^2} - \frac{m^2}{1 + m^2} - \frac{n^2}{1 + n^2} \pm \frac{2l}{\sqrt{1 + l^2}} \frac{m}{\sqrt{1 + m^2}} \frac{n}{\sqrt{1 + n^2}}\right)
\]

\[
= 1 - \cos^2 a - \cos^2 \beta - \cos^2 \gamma + 2 \cos a \cos \beta \cos \gamma
\]

\[
\sin^2 a \sin^2 \beta \sin^2 \gamma
\]

If \(l = \pm \cot \alpha, m = \pm \cot \beta, n = \pm \cot \gamma, \ A + \alpha = 180^\circ = 2a, \ B + b = 180^\circ = 2\beta, \ C + c = 180^\circ = 2\gamma, \) from (2).

In (8) let \(x = \cos \varphi, \) which is always possible, for from (1) we can show that \(x \) always lies between \(+1\) and \(-1\), and equation (8) becomes

(9) \[
\tan \varphi = \frac{2\sqrt{\cos \frac{1}{2}(a + \beta + \gamma) \cos \frac{1}{2}(\beta + \gamma - \alpha). \cos \frac{1}{2}(\alpha - \beta + \gamma) \cos \frac{1}{2}(a + \beta - \gamma)}}{\sin \alpha \sin \beta \sin \gamma}
\]

For the upper sign, by comparing equation (6) with (7) we see that the positive sign only is admissible.

If now we put \(a + \beta + \gamma = 2\omega \) this equation becomes

\[
\tan \varphi = \pm \frac{2\sqrt{\cos \omega \cos (\omega - \alpha) \cos (\omega - \beta) \cos (\omega - \gamma) \sin \alpha \sin \beta \sin \gamma}}{\sin \alpha \sin \beta \sin \gamma}
\]
which can easily be proved to be always possible. The other root will be
\[ \tan \varphi = \frac{+2\sqrt{\frac{\sin \omega \sin (\omega - a) \sin (\omega - \beta) \sin (\omega - \gamma)}}}{\sin \alpha \sin \beta \sin \gamma} \]
whether this value is real or not can be immediately inferred by inspection; it can be proved that \( \alpha, \beta, \gamma, \omega, (\omega - a), (\omega - \beta) \) and \( (\omega - \gamma) \) always lie between \( \pm 90^\circ \) when the sides and angles are each less than \( 180^\circ \).

Again from (2) \( \tan \frac{\omega}{2} (A - a) = lx = \cot \alpha \cos \varphi, \&c. \)
We have then \( A + a = 180^\circ \pm 2\alpha, B + b = 180^\circ \pm 2\beta, C + c = 180^\circ \pm 2\gamma, \alpha + \beta + \gamma = 2\omega; \)
\[ \tan \varphi = \frac{+2\sqrt{\frac{\cos \omega \cos (\omega - a) \cos (\omega - \beta) \cos (\omega - \gamma)}}}{\sin \alpha \sin \beta \sin \gamma} \]
which is always possible, or
\[ \tan \varphi = \frac{+2\sqrt{\frac{\sin \omega \sin (\omega - a) \sin (\omega - \beta) \sin (\omega - \gamma)}}}{\sin \alpha \sin \beta \sin \gamma} \]
\( \tan \frac{\omega}{2} (A - a) = \cos \varphi \cot \alpha, \tan \frac{\omega}{2} (B - b) = \cos \varphi \cot \beta, \tan \frac{\omega}{2} (C - c) = \cos \varphi \cot \gamma. \)
It will not make any difference whether we use \( A + a = 180^\circ + 2\alpha \) or \( A + a = 180^\circ - 2\alpha \&c., \) for the sign of \( \tan \varphi \) will correspondingly change; \( \tan \varphi \) and \( \cos \varphi \) must have the same sign.

121*.—"It has been cloudy during the last seven days; what is the probability that it will be cloudy to-morrow?"

SOLUTION BY PROF. H. T. J. LUDWICK, SALISBURY, N. C.

Let the probability that it will be cloudy on any particular day be denoted by \( x, \) then \( 1 - x \) is the probability that it will not be cloudy.

As \( x \) may vary from 0 to 1, and since it has been cloudy seven days
\[ \int_0^1 x^2 dx = \text{whole number of possible producing causes} \]
and \[ \int_0^1 x^2 dx = \text{" " favorable causes.} \]
Therefore the probability that it will be cloudy on the 8th day
\[ \frac{\int_0^1 x^2 dx}{\int_0^1 x^2 dx} = \frac{8}{9}. \]
122. "To inscribe a square in a given quadrilateral."

**Solution by Henry Heaton, B. S., Des Moines, Iowa.**

Let $ABCD$ be the given quadrilateral. At any convenient distance from the base, as $a$, draw $PQ$ parallel to the base and cutting $AD$ in $P$ and $BC$ in $Q$, and lay off $PS$ and $QR$ each equal to $a$. Draw $AS$ and prolong it to meet $BE$, drawn perpendicular to $AB$, in $E$. Draw $BR$ and prolong it to meet $AF$, drawn perpendicular to $AB$, in $F$, and join $EF$ cutting $DC$ in $G$. From $G$ draw $GO$ perpendicular to $AB$ cutting $B$-$F$ in $M$ and $AE$ in $L$, and draw $MK$ and $LH$ parallel to $AB$. Join $GH$ and $GK$ and draw $KI$ and $HT$ respectively parallel to $GH$ and $GK$; then is $GHIK$ the inscribed square required.

Because $LO$ in the triangle $EAB$ equals $GM$ in the triangle $EFB$, \( GL = MO \). By construction we have $BM : BR :: OM : a :: KM : a :: OM = KM$. Also, $AL : AS :: OL : a :: HL : a :: OL = HL$. Hence the right-angled triangles $GMK$ and $GLH$ are equal in all their parts and therefore $HKG$ is a right angle and $GH$ $IK$ is a square. Draw $IN$ perpendicular to $MK$; then is $IN = GL = MO$. Hence $I$ is on $AB$ and $GHIK$ is the required square.

If the line $EF$ should not intersect $CD$ it is evident there can be no solution; if it should coincide with it, there will be an infinite number.

Problem No. 62 of the Analyst may be constructed as follows:

Draw $BT$ and $AV$ in the given directions; $BT$ cutting $AD$ in $T$ and $AV$ cutting $BC$ in $V$, and join $TV$ cutting $CD$ in $G$. Then is $G$ the vertex of an angle of the required parallelogram.

123. "Solve the equation $y^2 = ax$ and determine what values of $a$ give real roots."

**Solution by Marcus Baker, U. S. Coast Survey, Washington, D. C.**

Taking the Napierian logarithms of both members we have

$$(\log x) + x = \log a = b.$$ 

Put $x = 1 - y$ and substitute for $\log x$ its value from the logarithmic series,

$$y + \frac{1}{2}y^2 + \frac{1}{3}y^3 + \frac{1}{4}y^4 + \frac{1}{6}y^6 + \cdots = -b = c,$$

or, performing the division,
\[ y + \frac{3}{2!} y^2 + \frac{11}{3!} y^3 + \frac{50}{4!} y^4 + \frac{274}{5!} y^5 + \frac{1764}{6!} y^6 + \frac{13068}{7!} y^7 + \ldots = c. \]

Reverting the series we have

\[ y = c - \frac{3}{2!} c^2 + \frac{16}{3!} c^3 - \frac{125}{4!} c^4 + \frac{1296}{5!} c^5 - \ldots. \]

\[ z = 1 - x = 1 - c + \frac{3}{2!}(\log a)^2 + \frac{4}{3!}(\log a)^3 + \ldots. \]

or

\[ z = 1 + \log a + \frac{3}{2!}(\log a)^2 + \frac{4}{3!}(\log a)^3 + \frac{5}{4!}(\log a)^4 + \frac{6}{5!}(\log a)^5 + \ldots. \]

If we now deduce the first differential coefficient of our expression \( z^x = a \), \( a \) being a variable, and equate to zero we find \( \log x = 1 \) or \( x = e \), the Napierian base; hence \( x \) has its greatest value when

\[ a = e^x = (2.71828)^{\frac{1}{x}} = 1.44467. \]

[Mr. Baker’s answer to Query 2 was overlooked in making up our notices for No. 4. He says the equation can be solved, and for proof sends two solutions by approximation. Such solution, however, was probably not contemplated by the querist, as all equations are thus solvable.]

124. "A radius is drawn in the circle \( x^2 + y^2 = a^2 \) and from its extremity an ordinate. From the foot of the ordinate a line is drawn perpendicular to the radius. Find and discuss the envelope of this last line."

SOLUTION BY HENRY HEATON, B. S., DES MOINES, IOWA.

Let \( PO \) and \( PR \) be two consecutive positions of the given line. Put \( CT = a \), \( CN = x \), \( TN = y \) and \( MN = dx \). From the similar triangles \( CTN \) and \( CNR \) we find \( CR = x^2/a \); \( RO = 2xdx/a \)

Through \( P \) draw \( BL \) perpendicular and \( CV \) parallel to \( PR \); then is \( VL = 2y^2 + a \) and \( VB = 2x^2 + a \); \( BL = 2a \).

\( BL \), being perpendicular to \( PR \), is normal to the required curve, and because the envelope of the normal to any curve is the evolute of that curve, \( KEJ \), the envelope of \( BL \), is the evolute of the required curve.

If \( x \), and \( y \), represent the coordinates of the point \( P \), we shall have
\[ x_r = x + \frac{x^3 y}{a^3} \ldots \ldots (1); \quad y_r = -\frac{x^3 y}{a^3}; \ldots \ldots (2) \]

By eliminating \( x \) and \( y \) from (1) and (2) we get
\[ 16a^3[x_r + \sqrt{(x_r^3 - 8y_r^3)}] = [3x_r + \sqrt{(x_r^3 - 8y_r^3)}]^3, \ldots \ldots (3) \]
which is the equation of the curve referred to rectangular coordinates.

From (1) and (2), \( dx_r = a^{-2}(3y^3 - a^3)dx \) and \( dy_r = a^{-2}(3y^3 - a^3)dy \).
Therefore, if \( z_r \) is the length of the required curve and \( z \), that of the corresponding arc of the circle, \[ dz_r = \frac{a^{-2}(3y^3 - a^3)}{a}dz. \ldots \ldots \ldots \ldots \ldots \ldots (4) \]

Put \( \rho = \) the radius of curvature of the envelope. Then, as the tangents of the circle and envelope, at their corresponding points, are parallel, it follows that \( \rho \propto dz_r : dz. \ldots \rho = a^{-1}(3y^3 - a^3) \). Hence, when \( \rho = 0 \), as at \( D, E, F, G \), \( y = aq_1/3, x_r = \pm \frac{1}{3}aq_1/6 \), and the curve intersects its evolute in the points \( D, E, F, G \), and consists of four branches, of which \( DAE \) is the involute of \( DJ \) and \( EJ \), \( DC \), the involute of \( DI \) and \( GI \), &c.

From (4) we have \( dz_r = a^{-2}(3y^3 - a^3)dx \) or, because the whole envelope is four times \( CD \) plus four times \( DA \),

\[ z_r = \frac{4}{a^2} \int_a^{\frac{a}{\sqrt{2}}} \frac{(3y^3 - a^3)}{\sqrt{(a^3 - y^3)}}dy + \frac{4}{a^2} \int_{\frac{a}{\sqrt{2}}}^a \frac{(3y^3 - a^3)}{\sqrt{(a^3 - y^3)}}dy. \]

If \( A \) is the whole area of the envelope, we have
\[ dA = y_rdx_r = a^{-2}(3y^3 - a^3)(a^3 - y^3)dy; \]

\[ \ldots A = \frac{4}{a^2} \int_0^a (3y^3 - a^3)(a^3 - y^3)dy = \frac{1}{3}a^3n. \]

The equation of the curve \( JEK \), is \( x^k + y^k = (2a)^k \). Its length is \( 3a \), \( JE = a \), and \( KE = 2a \). Being the envelope of \( BL \) it is easily described, and hence furnishes a convenient means of describing the required curve.

[Mr. Seitz has also given a very elegant solution of this problem. He employs polar coordinates and obtains the equation for the curve \[ \rho = \frac{8a \sin \theta}{\sqrt{[(1 - \sin \theta)(1 + 3 \sin \theta)] \pm \sqrt{[(1 + \sin \theta)(1 - 3 \sin \theta)]}} \]
and for length and area, respectively,
\[ s = 4 \int_0^{\frac{a}{\sqrt{2}}} \frac{\rho d \rho}{(\rho^2 - p^2)^{\frac{1}{2}}} + 4 \int_{\frac{a}{\sqrt{2}}}^a \frac{\rho d \rho}{(\rho^2 - p^2)^{\frac{1}{2}}} = a(4\sqrt{2} + \sin^{-1} \frac{1}{3}), \]
\[ A = 2 \int_{\frac{a}{\sqrt{2}}}^a \frac{\rho d \rho}{(\rho^2 - p^2)^{\frac{1}{2}}} - 2 \int_0^{\frac{a}{\sqrt{2}}} \frac{\rho d \rho}{(\rho^2 - p^2)^{\frac{1}{2}}} = \frac{1}{3}a^3n. \]
PROBLEMS.

125. SELECTED. — From a point $E$, in a square field $ABCD$, lines drawn to the corners $A$, $B$ and $C$ are found to be 50, 30 and 40 rods, respectively. Required the side of the field.

126. BY MARCUS BAKER, U. S. COAST SURVEY.—In a plane triangle are given the vertical angle $A$ and the bisectors $\beta$ and $\gamma$ of the base angles $B$ and $C$; determine the triangle.

127. By Prof. J. Scheffer.—Denoting the radii of the inscribed and circumscribed circles in and about a quadrilateral by $r$ and $R$, and the distance between the two centres by $h$, prove the relation:

$$\left(\frac{r}{R+h}\right)^2 + \left(\frac{r}{R-h}\right)^2 = 1.$$

128. By Henry Gunder.—Prove the identity of

$$\frac{n(n-1)(n-2)(n-3) \ldots 1}{(m+1)(m+2) \ldots (m+n+1)} \quad \text{and} \quad \frac{1}{m+1} - \frac{n}{m+2} + \frac{n(n-1)}{1.2(m+3)} - \ldots$$

$$+ (-1)^{n} \frac{1}{(m+n+1)}$$

$n$ being a positive integer.

129. By Christine Ladd.—Three lines through the vertices of a triangle meet in a point, $P$. Through the intersection of each with the opposite side a perpendicular to that side is drawn and these three perpendiculars meet in a point. Find the locus of $P$.

130. PHIL. HOGLAN.—Required the dimensions of an open cylindrical vessel of a given capacity so that the smallest possible quantity of metal shall be used in its construction, its thickness being already determined upon.

131. By R. J. Adcock.—If a body be impelled by the force of a fluid having the velocity $v_1$, and if the force of the impelling fluid to move the body be as $mv + nv^2$, where $v$ is the velocity of the fluid relative to the body, what must be the velocity of the body in order that the work performed on the body by the fluid, in a unit of time, may be a maximum?

132. By Prof. A. Hall.—Integrate the expression

$$\int \frac{x \, dx}{(x^2 + 8)(x^2 - 1)}.$$

133. By Henry Heaton.—When $x = 0$ find the value of

$$\frac{x - \sin x}{\tan x - x}.$$
134. By Artemas Martin.—Through a point taken at random in the surface of a circle, two chords are drawn, one at random and the other at right angles to the radius through that point; find the average area of the quadrilateral formed by the extremities of the chords.

135. By E. B. Seitz.—A point is taken at random in the surface of a given circle, and from it a line equal in length to the radius is drawn, so as to lie wholly in the surface of the circle. Find the chance that the line intersects a given diameter.

136. From De Morgan’s Budget of Paradoxes, by request of Dr. Nelson.

Suppose a planked floor with thin visible seams between the planks. Let there be a thin straight rod not so long as the breadth of the planks. This rod, being tossed up at hazard, will either fall quite clear of the seams or will lie across one seam. Prove that in the long run the fraction of the whole number of trials in which a seam is intersected, will be the fraction which twice the length of the rod is to the circumference of the circle having the breadth of the plank for its diameter.

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ERRATA.

On page 118, line 12 from bottom, for $x^0$, read $0^0$.

" " 121, " 5, the factor $\frac{a+x}{\log x}$, should be written as an exponent of $x$.

" " " 11, for 0 as the exponent of $v$ in the third member of the equation, read $v$.

" " 124, " 10, for $AD$, read $A'D$.

" " 136, " 14, for the factor $e$ under the sign of integration, read $\Delta e$.

" " " 15, for $d\Delta$, read $\Delta d$.

" " 137, " 5, for is plain, read, as is plain.

" " " 15, for lesser, read better.

" " " 32, the exponent $\theta^3$, under the sign of integration, should be minus.

" " 138, " 16, for found $a_1$, read found $= a_1$.

" " " 24, for $x$, read $X$.

" " 139, " 4, for $x_1$ integrations, read $x_1$th integration.

" " " 10, dele the exponent $-\infty$.

" " " 22, insert parenthesis at end of line.

" " 140, " 22, for $[X-f(a_1, a_2, \ldots)]^3$, read $[X-f(a_1, a_2, \ldots, a_n)]^2$.

" " 153, " 17, for $\frac{1}{3}x^3$, read $\frac{1}{3}x^2$.

" " " 3, from bottom, for $a^2x$, within the brackets, read $a^2x$.

" " " " " for $u_{1(0) - xy}^2$, read $u_{1(0) - xy}^2$. 
NEW INVESTIGATION OF THE LAW OF ERRORS OF OBSERVATION.

BY CHAS. H. KUMMEL, ASSISTANT U. S. LAKE SURVEY, DETROIT, MICH.

(Continued from page 140.)

5. If \( m \) observations are made of a quantity \( M \) which is a known function of \( n \) unknown quantities \( x, y, z, \ldots w \) we have a determinate problem if \( m = n \). However if \( m > n \) as we have supposed, the problem admits of an infinite number of contradictory systems, since each observation is liable to error. It is a logical necessity to remove these contradictions and this is done by applying corrections to the observed quantities \( M_1, M_2, \ldots M_m \), which may be \( A_1, A_2, \ldots A_n \), respectively. Since we have no rigorous condition to determine these corrections we may ask for the most probable system of corrections and this problem may be solved by the principles already explained.

Assuming the observations to be of equal precision, (if not, they may be reduced to the same precision by (26) if the precisions are given or some numbers proportional to them,) we have, if the probable error of observation is \( r \), the compound probability of the errors \( A_1, A_2, \ldots A_m \), to be committed in any order

\[
P = \varphi_0 e^{-\frac{\Delta^2}{2r^2}}[\Delta^2].
\]

If this probability shall be a maximum then

\[
[A]^0 = \text{a minimum,}
\]

if the subscript 0 denotes that the corrections already satisfy the condition.

Now let \( M \) be a linear function of \( x, y, z, \ldots w \), [if not we can prepare it as (34'')], or let \( M \) be of the form

\[
M = ax + by + cz + \ldots kw + k,
\]
then we have for each observation, replacing $M_1 = k$, $M_2 = k$, \ldots $M_n = k$

by $\mu_1$, $\mu_2$, \ldots $\mu_n$, and supplying corrections

\begin{align*}
  a_1 x + b_1 y + c_1 z + \ldots + l_1 w - \mu_1 &= A_1 \\
  a_2 x + b_2 y + c_2 z + \ldots + l_2 w - \mu_2 &= A_2 \\
  \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\
  a_n x + b_n y + c_n z + \ldots + l_n w - \mu_n &= A_n.
\end{align*}

By (45) we must have

\begin{align*}
  \frac{dA_1}{dx} + \frac{dA_2}{dy} + \ldots + \frac{dA_n}{dx} &= 0, \quad \text{or}
  \\
  \frac{dA_1}{dy} + \frac{dA_2}{dy} + \ldots + \frac{dA_n}{dy} &= 0
\end{align*}

but by (47)

\begin{align*}
  \frac{dA_1}{dx} &= a_1; \quad \frac{dA_2}{dx} = a_2 \ldots \frac{dA_n}{dx} = a_n \\
  \frac{dA_1}{dy} &= b_1; \quad \frac{dA_2}{dy} = b_2 \ldots \frac{dA_n}{dy} = b_n \\
  \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\
  \frac{dA_1}{dw} &= l_1; \quad \frac{dA_2}{dw} = l_2 \ldots \frac{dA_n}{dw} = l_n.
\end{align*}

These values substituted in (48) give

\begin{align*}
  a_1 A_1 + a_2 A_2 + \ldots + a_n A_n &= [a A] = 0 \\
  b_1 A_1 + b_2 A_2 + \ldots + b_n A_n &= [b A] = 0 \\
  \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\
  l_1 A_1 + l_2 A_2 + \ldots + l_n A_n &= [l A] = 0.
\end{align*}

Multiplying (47) in their order by $\alpha_1$, $\alpha_2$ \ldots $\alpha_n$ and adding the products, then also taking the sum of the products by $b_1$, $b_2$, \ldots $b_n$ and so on we obtain regarding (50)

\begin{align*}
  [a^2] x + [ab] y + [ac] z + \ldots + [al] w - [a \mu] &= [a A] = 0 \\
  [ab] x + [b^2] y + [bc] z + \ldots + [bl] w - [b \mu] &= [b A] = 0 \\
  [ac] x + [bc] y + [c^2] z + \ldots + [cl] w - [c \mu] &= [c A] = 0 \\
  \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\
\end{align*}
These equations which are called normal equations will give those determinate values of \( x, y, z \) which are rendered most probable by the observations.

Multiplying (47) in succession by \(-\mu_1, -\mu_2, \ldots, -\mu_m\) and adding the products we have also

\[
-([a\mu]x - [b\mu]y - [c\mu]z - \ldots - [l\mu]w + [\mu^2] = -[\mu A]
\]

and since the sum of the products by \( A_1, A_2, \ldots, A_m \)

\[
[aA]x + [bA]y + [cA]z + \ldots + [lA]w - [\mu A] = -[\mu A] = [\mu^2]_0
\]

we have

\[
(52) \quad [a\mu]x - [b\mu]y - [c\mu]z - \ldots - [l\mu]w + [\mu^2] = [\mu^2]_0.
\]

Having found \( x, y, z, \ldots, w \) by (51) we can find the sum of minimum squares of the corrections \([\mu^2]_0\) by forming also the quantity \([\mu^2]\). We have then by (23')

\[
\epsilon = \sqrt{\left(\frac{[\mu^2]_0}{m}\right)},
\]

and by (30) the probable error of the corrections

\[
r = \rho \sqrt{\left(\frac{2[\mu^2]_0}{m}\right)}.
\]

This formula is of course only approximately true and rests on the assumption that the system of corrections \( A_1, A_2, \ldots, A_m \) sufficiently near corresponds to the ideal case in article 2. This must be the case as nearly as possible since it is the most probable system of corrections.

6. We next determine the probable errors of \( x, y, z, \ldots, w \). It is plain that these unknown quantities must be some linear functions of \( \mu_1, \mu_2, \ldots, \mu_m \), so that we may assume

\[
(54) \quad x = a_1\mu_1 + a_2\mu_2 + \ldots + a_m\mu_m = [a\mu]
\]

\[
y = [b\mu]
\]

\[
z = [c\mu]
\]

\[
\ldots
\]

\[
w = [l\mu].
\]

The probable error of observation or that of \( \mu_1, \mu_2, \ldots, \mu_m \) being \( r \), we have by (43')

\[
(55) \quad \begin{align*}
    r_x &= r \sqrt{[a^2]} \\
    r_y &= r \sqrt{[b^2]} \\
    r_z &= r \sqrt{[c^2]} \\
    \ldots \ldots \\
    r_w &= r \sqrt{[l^2]}.
\end{align*}
\]
Substituting (54) into (47), (50) and (52) we have

\[(47') \quad a_1[\lambda \mu] + b_1[\beta \mu] + c_1[\gamma \mu] + \ldots + l_1[\kappa \mu] - \mu_1 = \Delta_1 \]
\[a_2[\lambda \mu] + b_2[\beta \mu] + c_2[\gamma \mu] + \ldots + l_2[\kappa \mu] - \mu_2 = \Delta_2 \]
\[\ldots \]
\[a_m[\lambda \mu] + b_m[\beta \mu] + c_m[\gamma \mu] + \ldots + l_m[\kappa \mu] - \mu_m = \Delta_m \]

\[(50')[a^\alpha][a \mu] + [ab][\beta \mu] + [ac][\gamma \mu] + \ldots + [al][\lambda \mu] - [a \mu] = 0 \]
\[[ab][a \mu] + [b^\alpha][\beta \mu] + [bc][\gamma \mu] + \ldots + [bl][\lambda \mu] - [b \mu] = 0 \]
\[[ac][a \mu] + [bc][\beta \mu] + [c^\alpha][\gamma \mu] + \ldots + [cl][\lambda \mu] - [c \mu] = 0 \]
\[\ldots \]
\[[al][a \mu] + [bl][\beta \mu] + [cl][\gamma \mu] + \ldots + [l^\alpha][\lambda \mu] - [l \mu] = 0. \]

\[(52') - [a \mu][a \mu] - [b \mu][\beta \mu] - [c \mu][\gamma \mu] - \ldots - [l \mu][\lambda \mu] - [\mu^2] = [\mathcal{A}]_0. \]

The coefficients \( a_1, a_2, \ldots, a_m; \beta_1, \beta_2, \ldots \) being independent of \( \mu_1, \mu_2, \ldots, \mu_m \) we have, resolving (50'),

\[(56) \quad [a^\alpha]a_1 + [ab]b_1 + [ac]c_1 + \ldots + [al]l_1 = a_1 \]
\[[a^\alpha]a_2 + [ab]b_2 + [ac]c_2 + \ldots + [al]l_2 = a_2 \]
\[\ldots \]
\[[ab]a_1 + [b^\alpha]b_1 + [bc]c_1 + \ldots + [bl]l_1 = b_1 \]
\[[ab]a_2 + [b^\alpha]b_2 + [bc]c_2 + \ldots + [bl]l_2 = b_2 \]
\[\ldots \]
\[[al]a_1 + [bl]b_1 + [cl]c_1 + \ldots + [l^\alpha]l_1 = l_1 \]
\[[al]a_2 + [bl]b_2 + [cl]c_2 + \ldots + [l^\alpha]l_2 = l_2 \]
\[\ldots \]

Taking sums of products by \( a_1, a_2, \ldots \) then by \( b_1, b_2, \ldots \) by \( c_1, c_2, \ldots \) and so on, we easily see that we must have

\[(57) \quad [aa] = 1; \quad [ba] = 0; \quad [ca] = 0; \ldots \quad [la] = 0 \]
\[[a\beta] = 0; \quad [b\beta] = 1; \quad [c\beta] = 0; \ldots \quad [l\beta] = 0 \]
\[\ldots \]
\[[a\lambda] = 0; \quad [b\lambda] = 0; \quad [c\lambda] = 0; \ldots \quad [l\lambda] = 1. \]

We have then, taking the sum of the products of (56) by \( a_1, a_2, \ldots \)

\[(58) \quad [a^\alpha][a^\alpha] + [ab][a\beta] + [ac][a\gamma] + \ldots + [al][a\lambda] = [aa] = 1 \]
\[[ab][a^\alpha] + [b^\alpha][a\beta] + [bc][a\gamma] + \ldots + [bl][a\lambda] = [b\alpha] = 0 \]
\[[ac][a^\alpha] + [bc][a\beta] + [c^\alpha][a\gamma] + \ldots + [cl][a\lambda] = [c\alpha] = 0 \]
\[\ldots \]
\[[al][a^\alpha] + [bl][a\beta] + [cl][a\gamma] + \ldots + [l^\alpha][a\lambda] = [l\alpha] = 0. \]
Multiplying the first equation of (51) by \([a^2]\), the second by \([a\beta]\), the third by \([a\gamma]\) ... the nth by \([a\lambda]\) and adding the products we have by virtue of (58)

\[(58) \quad x = [a^2][a\mu] + [a\beta][b\mu] + [a\gamma][c\mu] + \ldots + [a\lambda][l\mu].\]

In a similar manner we can determine \(y\) from the system

\[(58') \quad [a^2][a\beta] + [ab][b\beta] + [ac][c\beta] + \ldots [al][l\beta] = [a\beta] = 0,
\quad [ab][a\beta] + [b^2][b\beta] + [bc][c\beta] + \ldots [bl][l\beta] = [b\beta] = 1,
\quad [ac][a\beta] + [bc][c\beta] + [b^2][b\beta] + \ldots [cl][l\beta] = [c\beta] = 0,
\quad [al][a\beta] + [bl][b\beta] + [cl][c\beta] + \ldots [l^2][l\beta] = [l\beta] = 0.\]

\[(59') \quad y = [a\beta][a\mu] + [\beta\beta][b\mu] + [\beta\gamma][c\mu] + \ldots + [\beta\lambda][l\mu],\]

and so on for the other unknown quantities.

To determine now the probable error of observation or that of \(\mu_1, \mu_2, \ldots, \mu_n\) we have from (48'), omitting subscripts,

\[\mu = a[a\mu] + b[b\mu] + c[c\mu] + \ldots + l[l\mu] - \Delta.\]

We have then the relation of probable error by (48')

\[r^2 = (aa_1 + b\beta_1 + c\gamma_1 \ldots l\lambda_1)^2 + (aa_2 + b\beta_2 + c\gamma_2 \ldots l\lambda_2)^2 + \ldots + (aa_n + b\beta_n + c\gamma_n \ldots l\lambda_n)^2\]

\[= a^2[a^2] + ab[a\beta] + ac[a\gamma] + \ldots + al[a\lambda] + b^2[b^2] + b\beta_1[b\beta_2] + b\gamma_1[b\gamma_2] + \ldots + bl[b\lambda] + c^2[c^2] + c\gamma_1[c\gamma_2] + \ldots + cl[c\lambda] + \ldots + l^2[l^2].\]

Restoring subscripts and taking sum of all these equations we have

\[(60) \quad mr^2 = [a^2][a^2] + [ab][a\beta] + [ac][a\gamma] + \ldots + [al][a\lambda] + r^2 + mr^2\]
\[+ [ab][a\beta] + [b^2][b\beta] + [bc][c\beta] + \ldots + [bl][l\beta] + \ldots + [l^2][l^2] + \ldots + [a\lambda][a\lambda] + [b\beta][b\beta] + \ldots + [l\lambda][l\lambda].\]

\[= mr^2 + mr^2 + \Delta\] by virtue of (58), (58'),

We have then the remarkable relation

\[(61) \quad r = r\Delta \sqrt{\frac{m}{m-n}};\]

hence by (53)

\[(62) \quad r = p\sqrt{\frac{2[\Delta^2]}{m-n}}.\]

This value being used in (55) we have also the probable errors of the unknown quantities.
7. If the observations are of different precision we must reduce them first to the same precision according to (26). Usually this is done however by means of weights, which are numbers proportional to the square of the precision, or what is the same thing proportional to the reciprocal square of the probable error. Thus if in last article we assign to an observation the weight unity those of \( x, y, z, \ldots w \) are

\[
p_{x} = \frac{1}{[a^2]}, \quad p_{y} = \frac{1}{[b^2]}, \quad \ldots \quad p_{w} = \frac{1}{[\lambda^2]}.
\]

If then, as usual, weights are given we reduce the observations to the same precision if we multiply them by the square root of the weight and then proceed as in last article. Formula (62) gives then the probable error of that standard of observations to which they all have been reduced.

8. If there is only one unknown quantity \( z \), (47) becomes

\[
a_{1}z - \mu_{1} = A_{1} \\
a_{2}z - \mu_{2} = A_{2} \\
\ldots \ldots \ldots \\
a_{m}z - \mu_{m} = A_{m}.
\]

These equations being of equal precision by hypothesis the coefficients \( a_{1}, a_{2}, a_{m} \) may be understood to be ratios of the actually observed quantity to the quantity sought, for instance, as if the quantity had been magnified in different degrees or repeated a different number of times. We have then from the first of (51)

\[
x = [a\mu] + [a^2],
\]

and from the first of (58)

\[
[a^2] = 1 \div [a^2], \text{ or } p_{z} = [a^2].
\]

We have also the probable error of observation by (62)

\[
r = \rho \sqrt{\frac{2[A^2]}{m-1}};
\]

hence

\[
r_{z} = \rho \sqrt{\frac{2[a^2]}{(m-1)[a^2]}}.
\]

If however we place

\[
x_{1} = \mu_{1} + a_{1} \; ; \; \delta_{1} = A_{1} + a_{1} \\
x_{2} = \mu_{2} + a_{2} \; ; \; \delta_{2} = A_{2} + a_{2} \\
\ldots \ldots \ldots \ldots \ldots \\
x_{m} = \mu_{m} + a_{m} \; ; \; \delta_{m} = A_{m} + a_{m},
\]

we have the equations of condition

\[
x - x_{1} = \delta_{1} \quad \text{(precision } a_{1} \text{ or weight } a_{1}^2) \\
x - x_{2} = \delta_{2} \quad \text{(" } a_{2} \text{ " " } a_{2}^2) \\
\ldots \ldots \ldots \ldots \ldots \\
x - x_{m} = \delta_{m} \quad \text{(" } a_{m} \text{ " " } a_{m}^2).
We find after reducing to same precision

\[ x = \frac{[a^2 x]}{[a^2]} = \frac{[px]}{[p]} \]  

(71)

\[ p_x = [p] \quad \text{sum of weights of individual obs.} \]

(72)

\[ r = \rho \sqrt{\frac{2[p \rho^r]}{m - 1}} \]

(73)

\[ r_z = \rho \sqrt{\frac{2[p \rho^r]}{(m-1)[p]}}. \]

(74)

Suppose now that \( a_1 = a_2 = \ldots a_m \) or \( p_1 = p_2 = \ldots p_m \), then the weighted mean (71) becomes the arithmetic mean and we have

\[ x = [x] + m \]  

(71')

\[ p_x = m \]  

(72')

\[ r = \rho \sqrt{\frac{2[\rho]}{m - 1}} \]

(73')

\[ r_z = \rho \sqrt{\frac{2[\rho]}{(m-1)m}}. \]

(74')

9. The above short essay contains every essential principle of the method of least squares, though only a small part of the many interesting developments. I merely sketch out a method of treating the general problem of \( m \) observations for \( n \) unknown quantities. It consists in determining the probability of the system relatively to the unknown quantities.

We had (44)

\[ P = \varphi^m e^{-\frac{\rho^2}{r^2} [\Delta^2]}; \]

the probability of the system relatively to the unknown quantities is then

\[ II = \frac{P dx \, dy \, dz \ldots dw}{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P \, dx} \]

(75)

and that of one unknown quantity for instance \( w \)

\[ \varphi(w) = \frac{dw \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P \, dx}{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P \, dx} \]

(76)
EXTRACTION OF ROOTS BY LOGARITHMS.

BY ARTEMAS MARTIN, ERIE, PA.

Logarithms would afford the readiest method of computing the roots of numbers if we had Tables of sufficient extent.

Let it be required to find the $n^{th}$ root of $a$.

1. Put $r + x = \sqrt[n]{a}$, then $\log r = \frac{1}{n} \log a$, and $r$ is the number whose logarithm is $\frac{1}{n}$ of the logarithm of $a$. Common logarithms may be used in this case.

2. Put $r + x = \sqrt[n]{a}$, where $r$ is the integral part of the root; then, using Napierian logarithms,

$$\log (r + x) = \frac{1}{n} \log a.$$ 

Expanding $\log (r + x)$ and transposing $\log r$,

$$(\frac{x}{r}) - \frac{1}{2} (\frac{x}{r})^2 + \frac{1}{3} (\frac{x}{r})^3 - \frac{1}{4} (\frac{x}{r})^4 + \&c. = \frac{1}{n} \log a - \log r \ldots (1)$$

Reverting this series,

$$\frac{x}{r} = \left(\frac{1}{n} \log a - \log r\right) + \frac{\left(\frac{1}{n} \log a - \log r\right)^2}{1 \cdot 2} + \frac{\left(\frac{1}{n} \log a - \log r\right)^3}{1 \cdot 2 \cdot 3}$$

$$+ \frac{\left(\frac{1}{n} \log a - \log r\right)^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{\left(\frac{1}{n} \log a - \log r\right)^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \&c.$$ 

Example. — Required the one-hundredth root of 2.

Put $a = 2$, $r = 1$ and $n = 100$; then

$$100\sqrt[100]{2} = 1 + \frac{\log 2}{100} + \frac{(\log 2)^2}{1 \cdot 2 \cdot (100)} + \frac{(\log 2)^3}{1 \cdot 2 \cdot 3 \cdot (100)}$$

$$+ \frac{(\log 2)^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot (100)} + \&c.$$ 

The Napierian logarithm of 2 is 0.693147180559 + ,

$$.100\sqrt[100]{2} = 1.0069555499 + .$$
SOLUTION OF THE GENERAL EQUATION OF THE FIFTH DEGREE.

TRANSLATED BY DR. A. B. NELSON, DANVILLE, KENTUCKY.

(Continued from page 148.)

§ 8.

From the 5 unknown quantities \((y, v, z, u, t)\) of an equation of the 5th degree:

I. \(x^5 + bx^4 - cx^3 + dx - e = 0\)

\((2 \times 3 \times 4 \times 5) = 120\) solution-sums can be formed, of which, nevertheless, every five are fifth roots of the same dignant, e. g.: \((y + v' + z'' + u''' + t''')^5 = (y' + v'' + z''' + u'' + t')^5 = (y'' + v''' + z'' + u + t')^5 = (y''' + v'' + z' + u' + t)^5 = (y'''' + v + z + u' + t)^5\).

The equation for determining the solution-sums of the 5th degree will therefore have 120 dimensions; nevertheless, since 5 of its unknown quantities are roots of the same dignant, it will in strict significance be an equation of the 24th degree.

This equation can be found, that is to say, its coefficients can be given in rational functions of those of equation I.

I construct from equation I. one whose unknown quantities are the 5th powers of \(y, v, z, u\) and \(t\); thus

II. \(x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0\),

wherein: \(A = 5(b + e)\)

\(B = (b^5 - 5b^4d + 5b^3d^2 - 15bce + 5b^2e^2 + 5cde + 10e^2)\)

\(C = (-5b^5d + 5b^4e^2 + 5b^3d^2 - 15bce^2 + 10bde + e^2 + 10e^3)\)

\(D = (-5bce^2 + 5bde^2 + 5cde^2 - 5bde + d^2 + e)\)

\(E = e^3\),

and, considering that in this case \(a = 0\) and properly

\(A = (a^5 - 5a^4b + 5a^3b^2 + 5ab^3 - 5ad - 5be + 5e)\),

thus from these 6 equations the value of \(a\) will be given by a function of \(A, B, C, D, E\), and, indeed, by means of an equation of the 125th degree; which, nevertheless, in strict significance (for \(x^5\)) will be only an equation of the 26th degree.

The value of \(a^5\) is known to me by equation I. (here \(a = 0\)), and if I now divide the found equation of the 25th degree by \(x^5 - a^5\), there remains an equation of the 24th degree for \(x^5\). The \(5 \times 24 = 120\) values of this \(x\) are indeed the 120 solution-sums; for, if I consider equation II. in its
origin, its unknown quantities are $y, y', y'', y''', v, v', v'', v''', v'''$, etc.; $a$ thus always denotes the coefficient of one dimension in an equation which has $y, v, z, u, t$ as factors of its summands, and has each one of them combined with $\sqrt[4]{1}$ in such a manner that the product of all these summands will be equal to $e$. This condition the solution-sums alone fulfill.

Every solution-sum must therefore be represented by means of one irrational quantity, which stands under a radical of the 5th degree; this irrational quantity can admit of no more and no less than 120 significations (I call these its harmonic values), since, if I assume one of the solution-sums $= \sqrt[4]{F}$, the remaining 119 possible solution-sums are easily formed by changing the signs of the radical quantities occurring in this value.

It follows from this also that 24 of these solution-sums, namely, 24 which are not roots of the same dignant and whose value I group by combining the expression $\sqrt[4]{F}$ according to what is outside in the form $\sqrt[4]{F}$ (while in the radicals of this dignant $F$ remains rational), must receive the determination of their values by changes of sign in the radicals of this dignant.

These radicals can therefore be of such form only that 24 harmonic values are given by them; that is to say, only such roots can occur whose index is a factor of 24, and, particularly, none of these radical quantities (part of the dignant $F$) can have the index 5.

\section*{§. 9.}

Among those 24 solution-sums whose element $y$ is combined with the value $\sqrt[4]{1} = 1$, there cannot be two or more which are 5th roots of the same dignant.

I chose this circumstance as characteristic of those solution-sums which are related as roots of the same dignant.

The 24 solution-sums in which the summand $y$ occurs without the imaginary factor $\sqrt[4]{1}$, thus represent 24 exactly determined groups of solution-sums.

It is found always that among these 24 sums there are 4 which stand in such a relation that, added to $a$ and divided by 5, they express the unknown quantity $y$ (§. 7, V.).

Consequently in the 24 groups already described 6 series of 4 such groups are formed.

As characteristic of these 6 series I assume that two of their elements have the same form; thus, for example, $(y + v')$ occurs in all six.

Thus, for the 24 solution-sums with the element $y$, which are not related as roots of the same dignant, I have constructed the 6 following series or groups of the second class:
III. (1) \((y+v' +z'' +u'''' +t''')\) \hspace{1cm} (4) \((y+v' +z'' +u'''' +t''')\)
\((y+v'' +z'''' +u' +t'')\) \hspace{1cm} (5) \((y+v'' +z'''' +u' +t'')\)
\((y+v''' +z'' +u' +t'')\) \hspace{1cm} (6) \((y+v''' +z'' +u' +t'')\)
\((y+v''' +z'' +u' +t'')\) \hspace{1cm} (7) \((y+v''' +z'' +u' +t'')\)
\((y+v''' +z'' +u' +t'')\) \hspace{1cm} (8) \((y+v''' +z'' +u' +t'')\)

From the 6 characteristic sums:

IV. (1) \((y+v' +z'' +u'''' +t''')\) \hspace{1cm} (4) \((y+v' +z'' +u'''' +t''')\)
\((y+v'' +z'''' +u' +t'')\) \hspace{1cm} (5) \((y+v'' +z'''' +u' +t'')\)
\((y+v''' +z'' +u' +t'')\) \hspace{1cm} (6) \((y+v''' +z'' +u' +t'')\)
\((y+v''' +z'' +u' +t'')\)

I further construct groups of the 3d and 4th classes, by threes and by twos, as this is signified by their juxtaposition and superposition.

While I sought and found external marks of the first group arranged according to mutual fitness and of the second group (§. 7) constructed likewise upon internal grounds, I am now permitted to consider these marks as authorized for application respectively to the following groupings.

I therefore mention as the mark of the third group-formation, that the characteristic equations involved (IV. 1 to 6) have in common \((y+v'+z'')\), \((y+v'+z''')\), or \((y+v'+z''''\).

Consequently these equations group themselves by twos and by threes: the latter, if it is assumed that only such elements of the 6 equations in question belong to this same group by threes, which have no equal element but y and v'.

With this arrangement the operation is at once reduced to the finding of the possible permutations of y, v, z, u and t, since in all functions of y, v, z, u and t which enter rationally (especially in the symmetrical functions and their symmetrical parts) the possible permutations of the 5 unknown quantities are found again.
I assume that one of the 24 solution-sums, for example

I. \( \sqrt{5} F(1) = f(1) = (y+v'+x''+u'''+v',\sqrt{5}p+g)
\]
\[+\frac{1}{4}y(1-\sqrt{5}+p+g)\frac{1}{4}z(1-\sqrt{5}+p-g)
\]
\[= y-\frac{1}{4}(v+z+u+\eta)+\frac{1}{4}[(v+\eta)-(z+u)]\sqrt{5}+\frac{1}{4}[(v+u)-(z+\eta)p
\]
\[+\frac{1}{4}((v+z)-(u+\eta)g
\]
\[= \frac{1}{4}y+\frac{1}{4}[(v+\eta)-(z+u)]\sqrt{5}+\frac{1}{4}[(v+u)-(z+\eta)]p+\frac{1}{4}((v+z)-(u+\eta)g.
\]

I employ the auxiliaries:

\[(v+\eta)-(z+u) = a, (v+u)-(z+\eta) = b, (v+z)-(u+\eta) = d, \]

III. \( f(1) = \frac{1}{4}(5y+a,\sqrt{5}+b,p+d,\eta)
\]
\[f(2) = \frac{1}{4}(5y-a,\sqrt{5}-b,p+d,\eta)
\]
\[f(3) = \frac{1}{4}(5y-a,\sqrt{5}+b,p-d,\eta)
\]
\[f(4) = \frac{1}{4}(5y+a,\sqrt{5}+b,p-d,\eta)
\]
\[f(5) = \frac{1}{4}(5y+b,\sqrt{5}+a,p+d,\eta)
\]
\[f(6) = \frac{1}{4}(5y+b,\sqrt{5}+a,p-d,\eta)
\]
\[f(7) = \frac{1}{4}(5y-b,\sqrt{5}+a,p-d,\eta)
\]
\[f(8) = \frac{1}{4}(5y+b,\sqrt{5}+a,p+d,\eta)
\]
\[f(9) = \frac{1}{4}(5y-d,\sqrt{5}+a,p+d,\eta)
\]
\[f(10) = \frac{1}{4}(5y-d,\sqrt{5}+a,p-d,\eta)
\]
\[f(11) = \frac{1}{4}(5y-d,\sqrt{5}+a,p+d,\eta)
\]
\[f(12) = \frac{1}{4}(5y+d,\sqrt{5}+a,p-d,\eta)
\]

I multiply together these expressions by twos, and, indeed, those of the same group in which the same unknown quantities are combined with those values of \( \sqrt{5} \) whose product is \( +5 \); thus, for example:

\[f(1), f(4) = \frac{(5y+a,\sqrt{5})^2-(b,p-d,\eta)^2}{16}
\]

\[f(1) \cdot f(4) = \frac{5(5y+a)^2+b^2+d^2) + 2\sqrt{5}(5y+b,\sqrt{5}+b,d,\eta-b^2+d^2)}{16}
\]

\[= 4(y^2 + z^2 + u^2 + v^2) = -8b.
\]
IV. \[ f(1)(4) = \frac{1}{6}[-20\beta + \sqrt{5(5a_y + b_d, -d^2 + d^2)}] \]
\[ f(2)(3) = \frac{1}{6}[-20\beta - \sqrt{5(5a_y + b_d, -d^2 + d^2)}] \]
\[ f(5)(8) = \frac{1}{6}[-20\beta + \sqrt{5(5b_y + a_d, -d^2 + a^2)}] \]
\[ f(6)(7) = \frac{1}{6}[-20\beta - \sqrt{5(5b_y + a_d, -d^2 + a^2)}] \]
\[ f(9)(12) = \frac{1}{6}[-20\beta + \sqrt{5(5d_y + a_b, -a^2 + b^2)}] \]
\[ f(10)(11) = \frac{1}{6}[-20\beta - \sqrt{5(5d_y + a_b, -a^2 + b^2)}] \]
\[ f(13)(16) = \frac{1}{6}[-20\beta + \sqrt{5(5b_y + a_d, -a^2 + d^2)}] \]
\[ f(14)(15) = \frac{1}{6}[-20\beta - \sqrt{5(5b_y + a_d, -a^2 + d^2)}] \]
\[ f(17)(20) = \frac{1}{6}[-20\beta + \sqrt{5(5a_y + b_d, -d^2 + b^2)}] \]
\[ f(18)(19) = \frac{1}{6}[-20\beta - \sqrt{5(5a_y + b_d, -d^2 + b^2)}] \]
\[ f(21)(24) = \frac{1}{6}[-20\beta + \sqrt{5(5d_y + a_b, -b^2 + a^2)}] \]
\[ f(22)(23) = \frac{1}{6}[-20\beta - \sqrt{5(5d_y + a_b, -b^2 + a^2)}] \]

Now, since \( 5a_y = 4a_y + a_v + vz - u(t - u) \):

\[
\begin{align*}
(5a_y + b_d) &= 4(yz + yz + yz - yz + vz) \\
(-b^2 + d^2) &= 4(vz + vz - vz - vz) \\
(5b_y + a_d) &= 4(yz + vz + vz - vz + vz) \\
(-a^2 + d^2) &= 4(vz + vz - vz - vz) \\
(5d_y + a_b) &= 4(yz + vz + vz - vz + vz) \\
(-a^2 + b^2) &= 4(vz + vz - vz - vz)
\end{align*}
\]

whence there results:

V. \[ (5a_y + b_d, -b^2 + d^2) = 4[(yz + vz + vz + vz + vz) - (yz + vz + vz + vz + vz)] \]
\[ (5a_y + b_d, -b^2 + d^2) = 4[(yz + vz + vz + vz + vz) - (yz + vz + vz + vz + vz)] \]
\[ (5b_y + a_d, -a^2 + d^2) = 4[(yz + vz + vz + vz + vz) - (yz + vz + vz + vz + vz)] \]
\[ (5d_y + a_b, -a^2 + b^2) = 4[(yz + vz + vz + vz + vz) - (yz + vz + vz + vz + vz)] \]

From IV. and V.:

VI. \[ f(1)(4) = \frac{1}{5}[-5\beta + [(yz + vz + vz + vz + vz) - (yz + vz + vz + vz + vz)] \sqrt{5}] \]
\[ f(2)(3) = \frac{1}{5}[-5\beta - [(yz + vz + vz + vz + vz) - (yz + vz + vz + vz + vz)] \sqrt{5}] \]

with easy analogies for the products: \( f(5)(8) \), etc. (V.)

From VI. there further results:

VII. \[ f(1)(2)(3)(4) = \frac{1}{6}[(25\beta - 5[(yz + vz + vz + vz + vz) - (yz + vz + vz + vz + vz) + vz]) ] \]
or, since \[ ((yv+yt+vt+zu+ut)-(yz+yu+vu+vt+zt))^2 \]
\[ = (y^2v^2 \ldots 10t^2) + 2d + 2(y^2vt + y^2zu + y^2yz + y^2ut + y^2vt + x^2yt + x^2yu + u^2yc \\
+ u^2zt + c^2yu + c^2zu) - 2(y^2vz + y^2vu + y^2zt + y^2ut + y^2vt + v^2yt \\
+ v^2zu + z^2ye + x^2vt + x^2yu + x^2ut + u^2yz + u^2vt + u^2yt + u^2vic + c^2yz + c^2vu \\
+ c^2yu + c^2vu) \]
\[ = (b^2+12d) + 4(y^2vt + y^2zu + v^2yz + v^2ut + x^2yt + x^2yu + u^2vy + u^2zt + c^2yu \\
+ c^2vu), \]
there follows: \[ f(1)(2)(3)(4) = \frac{1}{4}[25b^2 - 5(b^2 + 12d) - 20(y^2vt \ldots 10t, \text{as above}), \]
or

**VIII.** \[ f(1)(2)(3)(4) = 5[b^2 - 3d - (y^2vt + y^2zu + v^2yz + v^2ut + x^2yt \\
+ x^2vu + u^2vy + u^2zt + c^2yu + c^2vu)], \]

and analogously:

- **(2)** \[ f(5)(6)(7)(8) = 5[b^2 - 3d - (y^2vz + z^2u + v^2yt + v^2zu + x^2ye \\
+ x^2vt + u^2yz + u^2vt + c^2yz + c^2vu)], \]
- **(3)** \[ f(9)(10)(11)(12) = 5[b^2 - 3d - (y^2vu + y^2zt + v^2yu + v^2zt + x^2yt \\
+ x^2vu + y^2yz + u^2vt + c^2ye + c^2zu)], \]
- **(4)** \[ f(13)(14)(15)(16) = 5[b^2 - 3d - (y^2vu + y^2vt + v^2yz + v^2ut + x^2yu \\
+ z^2vt + u^2yt + u^2yz + c^2ye + c^2zu)], \]
- **(5)** \[ f(17)(18)(19)(20) = 5[b^2 - 3d - (y^2vu + y^2vt + v^2yu + v^2zt + x^2yu \\
+ z^2ut + u^2vt + u^2yz + c^2yz + c^2vu)], \]
- **(6)** \[ f(21)(22)(23)(24) = 5[b^2 - 3d - (y^2vu + y^2ut + v^2yt + v^2zt + x^2yu \\
+ z^2vt + u^2vy + u^2zt + c^2yz + c^2vu)]. \]

I put the expression occurring in \( V \).

\[ (yv+yt+vt+zu+ut) = m^* \text{ thus } \]
\[ (yz+yu+vu+vt+zt) = b - m; \]

hence there follows from **VII.**

**IX.** \[ f(1)(2)(3)(4) = \frac{25b^2 - 5(2m - b)2}{4} = 5(b^2 - m^2 + mb). \]

*(To be continued.)*

*I will always use the symbol m with this signification only.*
REDUCTION OF THE PROBLEM OF THREE BODIES.

BY G. W. HILL, NYACK TURNPIKE, N. Y.

The object of this article is to find the three differential equations which virtually determine the sides of the triangle formed by the three bodies, bringing to ones aid all the known finite integrals of the problem.

Lagrange was the first to treat this question in his *Essai sur le Problème des Trois Corps* (Oeuvres, Tome VI, p. 227); but the formulæ lacking symmetry, his editor, Serret, has, in a note, supplied this and pointed out an important error into which Otto Hesse, who had investigated this subject, (Journal für die Mathematik, Band LXXIV) had fallen.

By adopting an orthogonal substitution, at the outset, for reducing the number of coordinates from nine to six, we can prevent the masses from entering the equations except through the potential function or its derivatives. In this way symmetry indeed appears to be lost, but there is so great a gain in condensation of the formulæ, that we can carry out some of the eliminations which the previous writers have been content only to indicate.

Let \( \xi, \eta, \zeta; \xi', \eta', \zeta'; \xi'', \eta'', \zeta'' \) be the rectangular coordinates of the masses \( m, m', m'' \), the expression for the living force will be

\[
T = m \frac{d\xi^2 + d\eta^2 + d\zeta^2}{2d^2} + m' \frac{d\xi'^2 + d\eta'^2 + d\zeta'^2}{2d^2} + m'' \frac{d\xi''^2 + d\eta''^2 + d\zeta''^2}{2d^2};
\]

and \( A, A', A'' \) being given by the equations

\[
\begin{align*}
A & = (\xi' - \xi'')^2 + (\eta' - \eta'')^2 + (\zeta' - \zeta'')^2, \\
A' & = (\xi - \xi')^2 + (\eta - \eta')^2 + (\zeta - \zeta')^2, \\
A'' & = (\xi - \xi'')^2 + (\eta - \eta'')^2 + (\zeta - \zeta'')^2,
\end{align*}
\]

the potential function

\[
\mathcal{Q} = \frac{m' m''}{A} + \frac{m m''}{A'} + \frac{m m'}{A''}.
\]

Without lessening the generality, the origin of coordinates can be put at the centre of gravity, when the principle of the conservation of this centre will furnish the equations

\[
\begin{align*}
\frac{m \xi}{} + m' \xi' + m'' \xi'' & = 0, \\
\frac{m \eta}{} + m' \eta' + m'' \eta'' & = 0, \\
\frac{m \zeta}{} + m' \zeta' + m'' \zeta'' & = 0.
\end{align*}
\]

By means of these relations three of the variables can be eliminated and the number thus reduced from 9 to 6. This transformation is most elegantly accomplished by putting
\[
\begin{align*}
\xi &= ax + \beta x', \\
\eta &= ay + \beta y', \\
\zeta &= az + \beta z', \\
\xi' &= a'x + \beta' x', \\
\eta' &= a'y + \beta'y', \\
\zeta' &= a'z + \beta' z', \\
\xi'' &= a''x + \beta' x', \\
\eta'' &= a''y + \beta'y', \\
\zeta'' &= a''z + \beta' z',
\end{align*}
\]
where \(a, a', a'', \beta, \beta', \beta''\) are six constants which may be so taken that they satisfy the five equations
\[
\begin{align*}
ma + m'a' + m''a'' &= 0, \\
m\beta + m'\beta' + m''\beta'' &= 0, \\
uma + m'\alpha' + m''\alpha'' &= 0, \\
uma^2 + m'\alpha'^2 + m''\alpha'^2 &= 1, \\
um\beta^2 + m'\beta'^2 + m''\beta'^2 &= 1.
\end{align*}
\]

The first two are necessary in order that the equations (1) may be satisfied; the third is adopted in order that nothing but squares of differential coefficients may occur in the transformed \(T\); and, evidently, the last two may be adopted without thereby diminishing the generality of the transformation.

These equations may be solved elegantly in the following manner. Put
\[
\sqrt{m} = k \sin \gamma \cos \varepsilon, \quad \sqrt{m'} = k \sin \gamma \sin \varepsilon, \quad \sqrt{m''} = k \cos \gamma;
\]
and adopt the four quantities \(\varphi, \varphi', \omega, \omega'\) such that
\[
\begin{align*}
\sqrt{m} \alpha &= \sin \varphi \cos (\omega + \varepsilon), \\
\sqrt{m} \beta &= \sin \varphi' \cos (\omega' + \varepsilon), \\
\sqrt{m'} \alpha' &= \sin \varphi \sin (\omega + \varepsilon), \\
\sqrt{m'} \beta' &= \sin \varphi' \sin (\omega' + \varepsilon), \\
\sqrt{m''} \alpha'' &= \cos \varphi, \\
\sqrt{m''} \beta'' &= \cos \varphi';
\end{align*}
\]
it is plain that the last two of equations (2) will be satisfied, and the first three take the forms
\[
\begin{align*}
\cos \gamma \cos \varphi + \sin \gamma \sin \varphi \cos \omega &= 0, \\
\cos \gamma \cos \varphi' + \sin \gamma \sin \varphi' \cos \omega' &= 0, \\
\cos \varphi \cos \varphi' + \sin \varphi \sin \varphi' \cos (\omega - \omega') &= 0.
\end{align*}
\]

Hence if the quadrantal spherical triangle \(ABC\) is constructed, and the arc \(AD = \gamma\), having any arbitrary orientation on the sphere, drawn, and \(BD\) and \(CD\) joined, we shall have \(\varphi = BD\), \(\varphi' = CD\), \(\omega = ADB\), \(\omega' = ADC\) as the general solution of the system of equations (2).

Then, after substitution,
\[
T = \frac{dx^2 + dy^2 + dz^2}{2ds^2} + \frac{dx'^2 + dy'^2 + dz'^2}{2ds^2}.
\]
And if we put
\[
\begin{align*}
f &= a - a', \\
g &= \beta - \beta', \\
v &= a^2 + y^2 + z^2,
\end{align*}
\]
\[
\begin{align*}
f' &= a' - a, \\
g' &= \beta' - \beta, \\
v' &= a'^2 + y'^2 + z'^2,
\end{align*}
\]
\[
\begin{align*}
f'' &= a'' - a, \\
g'' &= \beta'' - \beta, \\
v'' &= a''^2 + y''^2 + z''^2.
\end{align*}
\]
we have the expressions
\[ \Delta^2 = f^2 + 2fgv'' + g^2v', \]
\[ \Delta^3 = f^2v + 2f'g'v'' + g^3v, \]
\[ \Delta^4 = f^2v + 2f''g'v'' + g^3v'. \]

The equations of motion are now
\[ \frac{d^2x}{dt^2} = \frac{d\Omega}{dv}, \quad \frac{d^2y}{dt^2} = \frac{d\Omega}{dv}, \quad \frac{d^2z}{dt^2} = \frac{d\Omega}{dv}, \]
\[ \frac{d^2x'}{dt^2} = \frac{d\Omega'}{dv}, \quad \frac{d^2y'}{dt^2} = \frac{d\Omega'}{dv}, \quad \frac{d^2z'}{dt^2} = \frac{d\Omega'}{dv}. \]

Or, regarding \( \Omega \) as a function of \( v, v', v'' \),
\[ \frac{d^2x}{dt^2} = 2x\frac{d\Omega}{dv} + x'\frac{d\Omega}{dv}, \quad \frac{d^2x'}{dt^2} = 2x'\frac{d\Omega}{dv} + x\frac{d\Omega}{dv}, \]
\[ \frac{d^2y}{dt^2} = 2y\frac{d\Omega}{dv} + y'\frac{d\Omega}{dv}, \quad \frac{d^2y'}{dt^2} = 2y'\frac{d\Omega}{dv} + y\frac{d\Omega}{dv}, \]
\[ \frac{d^2z}{dt^2} = 2z\frac{d\Omega}{dv} + z'\frac{d\Omega}{dv}, \quad \frac{d^2z'}{dt^2} = 2z'\frac{d\Omega}{dv} + z\frac{d\Omega}{dv}. \]

From these, by eliminating the partial derivatives of \( \Omega \), we obtain
\[ \frac{x'd^2y - y'd^2x}{dt^2} + \frac{x'd^2y - y'd^2x'}{dt^2} = 0, \]
\[ \frac{zd^2x - xd^2z}{dt^2} + \frac{z'd^2x - z'd^2z'}{dt^2} = 0, \]
\[ \frac{yd^2z - zd^2y}{dt^2} + \frac{y'd^2z - z'd^2y'}{dt^2} = 0. \]

The integrals of which are
\[ \frac{xdy - ydx}{dt} + \frac{x'dy' - y'dx'}{dt} = k \cos \mu, \]
\[ \frac{x'dx - xdx}{dt} + \frac{z'dx' - z'dx'}{dt} = k \sin \mu \cos \nu, \]
\[ \frac{y'dx' - ydx}{dt} + \frac{y'dy' - y'dy'}{dt} = k \sin \mu \sin \nu, \]

\( k, \mu \) and \( \nu \) being the arbitrary constants. In addition there is the integral of living forces
\[ T = \Omega + h. \]

If we put
\[ u = \frac{dx^2 + dy^2 + dz^2}{dt^2}, \quad u' = \frac{dx^2 + dy^2 + dz^2}{dt^2}, \quad u'' = \frac{dx'dx' + dy'dy' + dz'dz'}{dt^2}, \]
it is evident that
\[ \frac{1}{2} \frac{d^2 v}{dt^2} - u = \frac{xd^2 x + yd^2 y + zd^2 z}{dt^2}, \]

\[ \frac{1}{2} \frac{d^2 v'}{dt^2} - u' = \frac{x'd^2 x' + y'd^2 y' + z'd^2 z'}{dt^2}, \]

\[ \frac{d^2 v''}{dt^2} - u'' = \frac{xd^2 x' + yd^2 y' + zd^2 z' + x'd^2 x + y'd^2 y + z'd^2 z}{dt^2}. \]

We put moreover

\[ \rho = \frac{xdx' - x'dx + ydy' - y'dy + zdx' - z'dx}{2dt}, \]

whence

\[ \frac{x'dx' + ydy' + zdx'}{dt} = \frac{1}{2} \frac{dv''}{dt} + \rho, \]

\[ \frac{x'dx + y'dy + z'dx}{dt} = \frac{1}{2} \frac{dv'}{dt} - \rho. \]

We have

\[ \frac{d\rho}{dt} = \frac{xd^2 x' - x'd^2 x + yd^2 y' - y'd^2 y + zd^2 z' - z'd^2 z}{2dt^2}. \]

By the substitution of the values of \( \frac{d^2 x}{dt^2}, \) &c., we obtain

\[ \frac{1}{2} \frac{d^2 v}{dt^2} - u = 2v \frac{d\Omega}{dv} + v' \frac{d\Omega}{dv'}, \]

\[ \frac{1}{2} \frac{d^2 v'}{dt^2} - u' = 2v' \frac{d\Omega}{dv'} + v'' \frac{d\Omega}{dv''}, \]

\[ \frac{d^2 v''}{dt^2} - u'' = 2v'' \left( \frac{d\Omega}{dv'} + \frac{d\Omega}{dv''} \right) + (v + v') \frac{d\Omega}{dv''}, \]

\[ \frac{d\rho}{dt} = v'' \left( \frac{d\Omega}{dv'} - \frac{d\Omega}{dv''} \right) + \frac{1}{2} (v - v') \frac{d\Omega}{dv''}. \]

These equations take simpler forms when the variables are changed as follows,

\[ w = \frac{1}{2} (v + v'), \quad w' = \frac{1}{2} (v - v'), \quad w'' = \frac{3}{2} v''. \]

\[ v = \frac{1}{2} (u + u'), \quad v' = \frac{1}{2} (u - u'), \quad v'' = u''. \]

Then they become

\[ \begin{cases} \frac{d^2 w}{dt^2} - \nu = \frac{d\Omega}{dw} + \frac{w}{d\Omega} + \frac{w''}{d\Omega} = -\frac{1}{2} \Omega, \\ \frac{d^2 w'}{dt^2} - \nu' = \frac{d\Omega}{dw'} + \frac{w'}{d\Omega} \frac{d\Omega}{dw''}, \\ \frac{d^2 w''}{dt^2} - \frac{3}{2} \nu'' = \frac{w''}{d\Omega} + \frac{w'''}{d\Omega} \\ \frac{d\rho}{dt} = w' \frac{d\Omega}{dw''} - w'' \frac{d\Omega}{dw''} \end{cases} \]

\[ \ldots (3) \]
If we add to the first of these the equation of living forces \( \nu = \Omega + \dot{h} \), we get

\[
\frac{d^2w}{dt^2} = \frac{1}{2} \Omega + \dot{h}, \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (4)
\]

an equation involving only the variables \( w, w' \) and \( w'' \).

If we square the members of the three equations which constitute the principle of conservation of areas, and take their sum, the result will evidently be an equation which is not changed by a cyclical permutation of the letters \( x, y, z \).

We have the identical equation

\[
(xdy - ydx)^2 + (xdx - xdy)^2 + (ydz - ydx)^2 = (x^2 + y^2 + z^2)(dx^2 + dy^2 + dz^2) - (xdx + ydy + zdz)^2,
\]

with the similar equation which is obtained by affixing accents to \( x, y, z \).

In addition there is the identity

\[
(xdy - ydx)(x'dy' - y'dx') + (xdx - xdy)(x'dx' - x'dy') + (ydz - ydx)(y'dz' - z'dy') = (xx' + yy' + zz')(xdx' + ydy' + zdz') - (xdx' + ydy' + zdz')(xx' + yy' + zz'),
\]

From these equations it will be seen that the equation of the sum of the squares takes the form

\[
u u + v' u' + 2 w' u'' - \frac{dv^2 + dv'^2 + dv''^2}{4dt^2} + 2 \rho^2 = k^2,
\]
or, after transforming into terms of the new variables, and for convenience writing \( k \) for \( \frac{1}{4}k \)

\[
\rho^2 = 2k^2 + \frac{dw^2 + dw'^2 + dw''^2}{dt^2} - 2(wv + w'v' + w''v''), \ldots \ldots (5)
\]
an equation which is symmetrical.

It is evident now that, since the values of \( \nu, \nu' \) and \( \nu'' \) are known from the first three equations of (3), we shall have, as the equations determining \( w, w' \) and \( w'' \), (4), (5) and the last of (3), provided we can find a relation connecting \( \rho \) with \( w, w', w'', \nu, \nu', \nu'' \) and the differentials of the first three.

Such a relation can be found in the following manner:—assume the four indeterminates \( X, X', X'', X''' \) so that the equations

\[
xX + x' X' + \frac{dx}{dt} X'' + \frac{dx'}{dt} X''' = 0,
\]

\[
yX + y' X' + \frac{dy}{dt} X'' + \frac{dy'}{dt} X''' = 0,
\]

\[
zX + z' X' + \frac{dz}{dt} X'' + \frac{dz'}{dt} X''' = 0,
\]

are satisfied; and treat the last as if they were equations of condition in the method of least squares, that is, multiply the first by \( x \), the second by \( y \),
the third by \( z \), and take the sum for a first equation; and so on. In this way the normal equations formed from them are

\[
\begin{align*}
vX + v''X' + \left( \frac{1}{2} \frac{dv}{dt} \right) X'' + \left( \frac{1}{2} \frac{dv''}{dt} + \rho \right) X''' &= 0, \\
v''X + v'X' + \left( \frac{1}{2} \frac{dv'}{dt} - \rho \right) X'' + \frac{1}{2} \frac{dv'}{dt} X''' &= 0, \\
\frac{1}{2} \frac{dv}{dt} X + \left( \frac{1}{2} \frac{dv}{dt} - \rho \right) X' + u X'' + u''X''' &= 0, \\
\left( \frac{1}{2} \frac{dv}{dt} + \rho \right) X + \frac{1}{2} \frac{dv}{dt} X' + u''X'' + u'X''' &= 0.
\end{align*}
\]

As the number of these equations exceeds that of those from which they were derived, they are not independent, and the determinant, formed from the coefficients, vanishes, which is the condition determining \( \rho \). This equation is

\[
\begin{align*}
\left[ \rho^2 + \frac{dv'dv'' - dv'''}{4d^2} \right]^2 + (v' - v''')(uv' - u''') &
\left[ v'\rho^3 + \frac{v'dv'' - v''dv'}{2d} - \rho + \frac{v'dv''^2 - 2v''dv'dv'' + v'dv''^2}{4d^2} \right] u \\
&\left[ v'\rho^3 + \frac{v'dv'' - v''dv'}{2d} + \frac{v'dv''^2 - 2v''dv'dv'' + v'dv''^2}{4d^2} \right] u' \\
+ 2 \left[ v'\rho^3 + \frac{v'dv'' - dv'dv'}{2d} + \frac{(v' - v'')(v'dv'' - v'dv'' + v'dv''^2)}{4d^2} \right] u'' &= 0,
\end{align*}
\]

or, expressed in terms of the new variables,

\[
\begin{align*}
\left[ \rho^2 + \frac{dw^2 - dw^2 + dw^2}{dt^2} \right]^2 + 4[w^2 - w'^2 - w''^2] [v' - v'' - v'''] &
\left[ w^{2} - w'^{2} - w''^{2} \right] \left[ v' - v'' - v''' \right] \\
-4[w - w' - w'' - w'''] \rho^3 &
+ 8 \left[ \frac{w'dw' - w'dw'}{dt} + \frac{w'dw - w'dw}{dt} \right] \left[ \frac{w'dw' - w'dw'}{dt} + \frac{w'dw - w'dw}{dt} \right] \rho \\
-4 \left[ \frac{w'dw^2 + dw'^2 + dw''^2}{dt^2} - \frac{dw d(w^2 + w'^2)}{dt \ dt} \right] \nu &
-4 \left[ \frac{w'dw^2 + dw'^2 + dw''^2}{dt^2} - \frac{dw d(w^2 - w'^2)}{dt \ dt} \right] \nu' \\
-4 \left[ \frac{w'dw^2 - dw'^2 + dw''^2}{dt^2} - \frac{dw d(w^2 - w'^2)}{dt \ dt} \right] \nu'' &\left[ \frac{w'dw^2 + dw'^2 - dw''^2}{dt^2} - \frac{dw d(w^2 - w'^2)}{dt \ dt} \right] \nu''' = 0.
\end{align*}
\]

If \( \rho \) is eliminated from this equation by means of its value from (5), we shall have an equation of the first degree in \( \rho \), from which the value of this quantity can be derived; this value will be written below.
In résumé, we can present our results as follows:—Let the five symbols, \( \Omega, \nu, \nu', \nu'' \) and \( \rho \), have the significations

\[
\Omega = [aw + a'w' + a''w'']' + [bw + b'w' + b''w'']' + [cw + c'w' + c''w'']',
\]

where \( a, b, c, \&c. \) denote certain functions of the masses and of a single arbitrary quantity,

\[
\begin{align*}
\nu &= \frac{d^2 w}{dt^2} - w \frac{d\Omega}{dw} - w' \frac{d\Omega}{dw'} - w'' \frac{d\Omega}{dw''} = \frac{d^2 w}{dt^2} + \frac{1}{2} \frac{d\Omega}{dw}, \\
\nu' &= \frac{d^2 w'}{dt^2} - w' \frac{d\Omega}{dw} - w'' \frac{d\Omega}{dw''}, \\
\nu'' &= \frac{d^2 w''}{dt^2} - w'' \frac{d\Omega}{dw} - w'' \frac{d\Omega}{dw''}.
\end{align*}
\]

\[
\rho = 2 \left[ \frac{d^2 w}{dt^2} - w \frac{d\Omega}{dw} + \frac{1}{2} \frac{d\Omega}{dw} \right]
\]

Then the differential equations, which determine \( w, w' \) and \( w'' \), are

\[
\begin{align*}
\frac{d^2 w}{dt^2} &= \frac{1}{2} \frac{d\Omega}{dw} + h, \\
\rho^2 &= 2k^2 + \frac{d^2 w^2 + d^2 w'^2 + d^2 w''^2}{dt^2} - 2(wu + w'u' + w''u''), \\
\frac{d\rho}{dt} &= w' \frac{d\Omega}{dw} - w'' \frac{d\Omega}{dw''}.
\end{align*}
\]

The first and second are of the second order, while the third is of the third order. It will be noticed that, although the expressions, involved in them, are not exactly symmetrical, yet they exhibit some approach to symmetry; and perhaps by a slight change in the variables they may be made so. But I have not succeeded in discovering such a transformation.

*For want of sorts the Greek \( \nu \) is here, and throughout the remainder of this article, written for \( \psi \).—Composer.
Remark. — If the gravitating force requires time for its transmission through space, it can be easily proved, by the well known principle of the parallelogram of velocities and forces, that every planetary body must be accelerated in its orbit, and, consequently, must recede from the gravitating center in an outward spiral path, unless such propulsion is counteracted by a resisting medium.

Problem I. — Assuming that every particle of matter in the universe transmits its gravitating force to every other particle with the velocity of the waves of light,* heat, electricity, and other solar radiations, that is, at the rate of 186420 miles per second; prove that the propelling and resisting forces of the planets, in circular orbits, must be directly as the masses, and inversely as the fifth powers of the square roots of the respective distances from the sun.

Problem II. — When the two antagonistic forces, in circular orbits, are equal, what must be the law of density of the resisting ethereal medium, expressed in terms of the sun’s distance?

Problem III. — Given the earth’s mean radius equal to 3955.94943182 miles; distance from the sun = 91430000 miles; sun’s mass (that of the earth = 1) = 314760; earth’s density (that of water = 1) = 5.6604; weight of a cubic inch of distilled water = 252.458 grains avoirdupois; one pound, when freed from the effects of centrifugal force of the earth’s axial rotation, and from atmospheric influences, = 7000 grains; the velocity of gravity as in problem I, — to find the intensity of the orbital and resisting forces, expressed in pounds weight, to permanently maintain the earth in a circular orbit.

Problem IV. — Assuming the velocity of gravity as in problem I, and the equality of the two forces as in problem II, — prove that the aberrating forces, in any two points of an elliptic orbit, vary inversely as the cubes of their distances from the center of gravity, situated in the lower focus of the ellipse.

*Laplace concludes, from a consideration of the moon’s secular equation, that, if gravitation is produced by the impulse of a fluid directed towards the center of the attracting body, the velocity of the gravitating fluid must be, at least, a hundred millions of times greater than that of light. — See Mécanique Celeste, Bowditch’s translation, Vol. IV, p. 645.—Ed.
Problem V. — Assuming the same conditions as in problem IV, prove that in any two points of an elliptic orbit, the resistance will vary directly as the distance to the upper focus, and inversely as the fifth power of the square root of the distance to the lower focus.

Problem VI. — If the two forces are equal in circular orbits; and if $a$ is equal to the semi-major axis of the earth's orbit; and $v, g$ and $r$ are respectively equal to the orbital velocity, intensity of the orbital force of gravity, and the etherial resistance of the earth at its mean distance from the sun; and if $a' = \text{the semi-major axis, and } b' = \text{the semi-minor axis of any elliptic orbit},$—prove that one of the values of $x,$ in the equation,

$$x^2 - 4a'x^2 + 4a'^2x - r^2g^2b'^2a'^2 + r^2 = 0,$$

will be equal to the length of the radius vector to that point in the ellipse, where the accelerating force will be exactly balanced by the resisting force.

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NOTE BY PROF. M. C. STEVENSON.—In the article on Repetends published in Nos. 1 and 2 of Vol. I, it is stated, p. 25, lines 7 to 10, that “the operation may be very much abbreviated by taking advantage of the well-known property of repetends, that after one half of the figures are obtained, the second half may be found by subtracting each figure of the first half successively from 9;” and in demonstrating this property, on p. 27, it is assumed, if $1+a$ be a fraction that reduces to a repetend, that in the reduction by division there will occur the remainder $d - 1.$

It should have been stated that this property applies only to such fractions reducing to pure repetends as have for denominators prime numbers.

It is therefore evident that the property is not applicable to the repetends resulting from $\frac{1}{17}, \frac{1}{19}, \frac{1}{19}, \&c.$

The restriction is implied in the proof, since the remainder $d - 1$ only occurs in case the denominator is a prime number.

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NOTE BY THE EDITOR. — The solution of problem 105, given at p. 127, is defective, because the conclusion is virtually assumed by placing $q - p = m - A + B - C + D - \&c.$ If we write instead, $p - q = -m + A - B + C - D - \&c.,$ we shall prove, in a similar manner, that $p > q.$

We subjoin Dr. Nelson's solution of 105, which is entirely rigorous.

“The number of odd selections (1 at a time, 3 at a time, &c.) out of $n$ shot

$$S = n + \frac{n(n-1)(n-2)}{3!} + \frac{n(n-1)(n-2)(n-3)(n-4)}{5!} + \&c.$$
The number of even selections (2 at a time, 4 at a time, &c.)
\[ S' = \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)(n-3)}{4!} + \&c. \]
Now \( (1-1)^n = 0 \)
\[ = 1 - n + \frac{n(n-1)}{2!} - \frac{n(n-1)(n-2)}{3!} + \frac{n(n-1)(n-2)(n-3)}{4!} + \&c. \]
The sum of the negative terms in this series \( = S \), and the sum of the positive terms \( = S' + 1 \). Hence \( S = S' + 1 \), or \( S - S' = 1 \). That is, the possible odd selections exceed the possible even selections, and therefore the chances are in favor of drawing an odd number."

**SOLUTION OF PROBLEMS IN NUMBER FIVE.**

Solutions of problems in number five have been received as follows:
From R. J. Adecock, 136; Marcus Baker, 125, 126, 127, 130 and 133; Henry Gunder, 125, 126, 127, 128, 130, 133 and 136; William Hoover, 125, 126, 130, 133 and 136; Henry Heaton, 125, 128 and 135; Christine Ladd, 129; Prof. H. T. J. Ludwick, 130, 135 and 136; Artemas Martin, 125, 133, 134 and 136; D. J. Mc. Adam, 125, 130 and 133; E. B. Seitz, 125, 127, 129, 130, 133, 134, 135 and 136; Prof. J. Scheffer, 125, 126, 127, 128, 129, 130 and 133.

125.—"From a point \( E \), in a square field \( ABCD \), lines drawn to the corners \( A, B \) and \( C \) are found to be 50, 30 and 40 rods, respectively. Required the side of the field."

**SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.**

Construct the right angled isosceles triangle \( BEF \), making \( BF = BE = b = 30 \) rods; on \( EF \) construct the triangle \( AEF \), making \( AE = a = 50 \) rods, and \( AF = c = 40 \) rods; and on \( AB \) construct the square \( ABCD \), which is the square required; for we can easily prove that \( EC = AF \).

Put \( \angle AEF = a \). Then we have \( \cos a = \frac{(a^2 + 2b^2 - c^2)}{(2ab \sqrt{2})} \), and \( AB = \sqrt{[a^2 + b^2 - 2ab \cos (a + \frac{1}{2}\pi)]} \)
\[ = \frac{1}{2} \sqrt{[2a^2 + 2c^2 + 2 \sqrt{(4a^2b^2 + 4b^2c^2 + 2a^2c^2 - 4b^2 - a^2 - c^2)}]} \]
\[ = 5 \sqrt{(82 + 6 \sqrt{119})} = 60.71496 \) rods.

126.—"In a plane triangle are given the vertical angle \( A \) and the bisectors \( \beta \) and \( \gamma \) of the base angles \( B \) and \( C \); determine the triangle."
We easily obtain the proportions, \( c : \beta = \sin(A + \frac{1}{2}B) : \sin A; \ b : \gamma = \sin(A + \frac{1}{2}C) : \sin A \). But since \( c : b = \sin C : \sin B \) we get

\[
\frac{\beta \sin(A + \frac{1}{2}B)}{\gamma \sin(A + \frac{1}{2}C)} = \frac{\sin C}{\sin B} = \frac{\sin(A + B) \cos \frac{1}{2}(A + B)}{\sin \frac{1}{2}B \cos \frac{1}{2}B}.
\]

But \( \sin(A + \frac{1}{2}C) = \cos \frac{1}{2}(A - B) \), \( \ldots \). \( \beta \sin(A + \frac{1}{2}B) \sin \frac{1}{2}B \cos \frac{1}{2}B = \gamma \cos \frac{1}{2}(A - B) \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A + B) \), or \( \beta \sin \frac{1}{2}B \cos \frac{1}{2}B(\sin A \cos \frac{1}{2}B + \cos \frac{1}{2}A \sin \frac{1}{2}B) = \gamma (\cos \frac{1}{2}A \cos \frac{1}{2}B + \cos \frac{1}{2}A \cos \frac{1}{2}B - \sin \frac{1}{2}A \times \sin \frac{1}{2}B) \), or, dividing by \( \cos \frac{1}{2}B \cos \frac{1}{2}B \), and putting, for brevity, \( \frac{1}{2}A = m, \frac{1}{2}A = n \) and \( \tan \frac{1}{2}B = x \), we obtain, after reduction, the cubic eq'n,

\[
x^3 + \frac{(n^2 - m^2)r}{m^2}x + \frac{2m^3 - m^2n}{m^2}x + \frac{n}{m} = 0.
\]

127. "Denoting the radii of the inscribed and circumscribed circles in and about a quadrilateral by \( r \) and \( R \), and the distance between the two centres by \( h \), prove the relation:

\[
\left(\frac{r}{R+h}\right)^2 + \left(\frac{r}{R-h}\right)^2 = 1.
\]

SOLUTION BY E. B. SEITZ.

Let \( M, N \) be the centres of the inscribed and circumscribed circles. Put

\[
AB = a, BC = b, CD = c, DA = d, MF = r, \quad NB = R, MN = h, \quad \angle ABC = B.
\]

Then we have area \( ABCD = \sqrt{abcd} \), \( BG = \frac{1}{2}a \), \( BH = \frac{1}{2}b \), \( BF = R \cot \frac{1}{2}B \), \( NG = \frac{1}{2}b \csc B - \frac{1}{2}a \cot B \), \( r = \sqrt{abcd} \), \( \sin B = \frac{2\sqrt{abcd}}{ab+cd} \), \( \cos B = \frac{ab-ac}{ab+cd} \cot \frac{1}{2}B = \sqrt{\frac{ab}{cd}} \), \( a + c = b + d \) \ldots \ldots \ldots (1)

\[
\left(\frac{1}{2}c \csc B - a \cot B\right)^2 + \frac{1}{4}a^2 = R^2 \ldots \ldots \ldots \ldots \ldots \ldots (2)
\]

and \( \left(\frac{1}{2}b \csc B - \frac{1}{2}a \cot B - r\right)^2 + (r \cot \frac{1}{2}B - \frac{1}{2}a)^2 = h^2 \ldots \ldots \ldots (3) \)

Subtracting (2) from (3), substituting the value of \( \cot \frac{1}{2}B \), and reducing, we find \( R^2 + r^2 - h^2 = r^2 \left[b \csc B - a \cot B + (ar+c)\right] \ldots \ldots \ldots \ldots \ldots \ldots (4) \)

Squaring (4), adding (2) \( \times 4r^2 \), transposing and reducing, we have

\[
(R^2 + r^2 - h^2)^2 = 4R^4 + r^2 - a^2 + \frac{(a^2 - c^2)r^2}{c^2} + \frac{2abr}{c \sin B} - \frac{2a^2r \cos B}{c \sin B} \ldots \ldots \ldots (5)
\]

By substituting for \( r, \sin B \) and \( \cos B \), in the last three terms in (5), we have

\[
\frac{abc(a-c)}{c(a+c)} + \frac{ab(ab+cd)}{c(a+c)} - \frac{a^2(ab-cd)}{c(a+c)} = \frac{a^2}{c(a+c)} \left[b(b+d) - ab+cd\right] = a^2.
\]

Hence (5) reduces to \( (R^2 + r^2 - h^2)^2 = 4R^4 + r^2 \ldots \ldots \ldots \ldots \ldots \ldots (6) \)
By reduction (6) becomes \((R^a - h^b)^2 = 2r^a(R^a + h^b) = r^a(R + h)^a + r^a(R - h)^a\).

Dividing this equation by \((R^a - h^b)^2\), we get

\[
\left(\frac{r}{R + h}\right)^2 + \left(\frac{r}{R - h}\right)^2 = 1.
\]

128. "Prove the identity of

\[
\frac{n(n-1)(n-2)(n-3) \ldots 1}{(m+1)(m+2) \ldots (m+n+1)} \quad \text{and} \quad \frac{n}{m+1} - \frac{n}{m+2} + \frac{n(n-1)}{1.2(m+3)} - \ldots + (-1)^n \frac{1}{(m+n+1)},
\]

\(n\) being a positive integer."

**SOLUTION BY HENRY GUNDER NORTH MANCHESTER, IND.**

If the integral, \(\int_0^1 x^n(1-x)^m dx\), be taken by parts, and again by expanding \((1-x)^m\) in a series and integrating each term, the results are respectively the two members of the given equation; hence their equality.—See Todhunter’s Integral Calculus, pp. 70 and 71.

[We have an elegant solution of 129, by E. B. Seitz, but, for want of room, are compelled to defer its publication to No. 1, Vol. IV.]

130. "Required the dimensions of an open cylindrical vessel of a given capacity so that the smallest possible quantity of metal shall be used in its construction, its thickness being already determined upon."

**SOLUTION BY MARCUS BAKER, U. S. COAST SURVEY.**

Put \(r, h, c\) and \(t\) respectively equal the internal radius, internal height, capacity, and thickness of metal. Then is \(c = \pi r^2 h\), whence \(h = c - \pi r^2\).

The amount of metal used is \(\pi(r + t)^2h + t - c = c\) a minimum, or

\[(r + t)^2[(c - \pi r^2) + t^2] = [1 + (t - r)]^3[(c - \pi) + tr^2] = a\) minimum.\]

Differentiating, \(2[1 + (t - r)]^3r - 2[(c - \pi) + tr^2] [1 + (t - r)]t + r^2 = 0; \ldots r^2[1 + (t - r)] - (c - \pi) - tr^2 = 0; \ldots r = \phi'(c - \pi) = c + [\pi \phi'(c - \pi)^2] = h.\)

131. "If a body be impelled by the force of a fluid having the velocity \(v_1\), and if the force of the impelling fluid to move the body be as \(mv + nv^2\), where \(v\) is the velocity of the fluid relative to the body, what must be the velocity of the body in order that the work performed on the body by the fluid, in a unit of time, may be a maximum?"
SOLUTION BY HENRY HEATON.

Let \( x = \) the velocity of the body. Then the work performed is a max. when \( x(mv + nv^2) \) is a maximum. \( v = v_1 - x; \)

\[
\frac{du}{dx} = 3nx^2 - 2(2n + m)x^2 + (mv_1 + nv_1^2) = u.
\]

\[
\frac{d^2u}{dx^2} = 6nx^2 + m - (2n + m). \quad \cdots \cdots \cdots \cdots \cdots (1)
\]

From (1), \( x = [2n + m \pm \sqrt{(n v_1^2 + mn v_1 + m^2)}] \div 3n. \) Hence for a max. or a minimum, \( x = \) respectively, \( [2n + m \pm \sqrt{(n v_1^2 + mn v_1 + m^2)}] \div 3n. \)

132. "Integrate the expression \( \frac{x}{(x^3 + 8)\sqrt{(x^3 - 1)}}, \)"

[No solution of this problem has been received. The expression \( xdx \)

\[
\frac{xdx}{(x^3 + 8)\sqrt{(x^3 - 1)}} \]

may be written \( \frac{xdx}{\sqrt{[(x^3 + 8)(x^3 - 1)]}} = \frac{dx}{\sqrt{\varphi x}}, \) where \( \varphi x \) is a rational function of \( x. \) Hence, because the dimension of \( \varphi x \) is more than 2, and may be represented by \( 2n + 1, \) the given expression is an ultra elliptic function, of the \( n \)th class, and may be integrated by Abel's Theorem.—Ed.]

133. "When \( x = 0 \) find the value of \( \frac{x - \sin x}{\tan x - x}. \)"

SOLUTION BY D. J. MC. ADAM, WASHINGTON, PA.

Taking differential coefficients of both numerator and denominator, we get \( \frac{\cos^2 x}{1 + \cos x}, \) which, when \( x = 0, \) is \( \frac{1}{2}. \)

[Solutions of 134, 135 and 136, will be published in No. 1, Vol. IV, as follows: 134, by Artemas Martin; 135, by E. B. Seitz; 136, by W. Hoover.]

PROBLEMS.

137. By L. Regan.—A point \( D, \) is given in position between two lines which make a given angle at \( A. \) Find the position of a given line, \( BC, \) drawn through \( D, \) and intersecting the two lines in the points \( B \) and \( C. \)

138. By Prof. J. Scheffer.—In a pentagon, \( ABCDE, \) the triangles \( ABE = a, ADE = b, CDE = c, BCD = d, ABC = e \) are known; to find the area \( F \) of the pentagon.
139. **By Henry Gudr.**—Let $e$ be the eccentricity, or ratio of polar compression to the equatorial radius of the earth. Find the average length of the earth's radius.

140. **By Prof. D. Trowbridge.**—A point is taken at random within a given circle, and a random chord drawn through it. If another chord be drawn at random, what is the probability that it will intersect the first?

**Query.** **By Henry Guder.**—In Thompson and Tait's "Elements of Natural Philosophy" it is stated that, "at the southern base of a hemispherical hill radius $a$ and density $\rho$, the true latitude (as measured by the aid of the plumb-line) is diminished by the attraction of the mountain by the angle $\frac{3}{2}\rho a \div (G - \frac{1}{2}\rho a)$, where $G =$ the attraction of the earth in same units." How is this proved?

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**BOOK NOTICES.**

**The Cone and its Sections** treated geometrically, by S. A. Renshaw, of Nottingham.

In this valuable treatise on the Conic Sections, the author has mainly derived the properties of the curves from the cone, and has shown that they pertain alike both to the scalene and right cone.

The Geometrical method of demonstration is chiefly employed, and the student has the invaluable aid of elaborate diagrams, which enable him readily to grasp and hold the line of argument pursued. The value of the book, for students, is also greatly enhanced by the variety of examples it contains.—Students of this beautiful and interesting branch of geometry will surely be interested and instructed by a perusal of this book.

**New and Easy Method of Solution of the Cubic and Biquadratic Equations;** by Orson Pratt.
London. 151 pp. 8vo.

We are not at present prepared to speak of the merits of this book, but insert a single paragraph from the author's preface which will indicate its character.

"The author's discovery of the Equation of Differences, together with several other kindred discoveries, resulting from the properties of this equation, has enabled him to entirely dispense with every process for finding the limits of the roots; to dispense with the theorem of Sturm, and all similar theorems, having for their object the determination of the number of real roots and their situation in the arithmetical scale; to dispense with all processes for finding the first figure of a root by trial or successive substitution; and to dispense with the successive trial divisors used by Horner."

**The Wittenberger;** a monthly periodical, published at Springfield Ohio, has a Mathematical department, edited by Wm. Hoover of Bellefontaine, Ohio.

**Correction.**—On p. 155, Vol. III, Eq. (4), leave off the numerals 1.2, 1.2.3, &c., in the denominators. The four following equations are also wrong, but they can now easily be corrected. The error was made by me in copying. D. T.
THE ANALYST.

A JOURNAL OF

PURE AND APPLIED MATHEMATICS.

EDITED AND PUBLISHED BY

J. E. HENDRICKS, A. M.

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SINGULAR SOLUTIONS OF DIFFERENTIAL EQUATIONS OF THE FIRST ORDER.

BY PROF. W. W. JOHNSON, ST. JOHN'S COLLEGE, ANNAPOlis, MD.

1. A DIFFERENTIAL equation of the first order is usually written in the form
   \[ \varphi(x, y, p) = 0, \] \[ p = \frac{dy}{dx}. \]
   Regarding \( x \) and \( y \) as the rectangular coordinates of a moving point, \( p = \tan \varphi \), if \( \varphi \) denote the inclination of the point's motion to the axis of \( x \). The relation (1) must therefore be regarded not as the equation of a curve, or relation between the coordinates of a point, but rather as a relation between the position of a moving point and the direction of its motion. Thus the point \( (x, y) \) may have any assumed position as \( (a, b) \), and the corresponding value of \( p \) is any one of the roots of
   \[ \varphi(a, b, p) = 0. \]

Starting from the assumed position \( (a, b) \) the point may move in the direction assigned by one of the roots of (2); and, as the values of \( x \) and \( y \) vary, the value of \( p \) will in general vary; the point describing a curve. Different assumed initial positions will determine other curves, and we may say that a point satisfies the differential equation (1), provided it is moving in any one of the system of curves thus generated.

2. The general equation of this system of curves is the Complete Primitive of (1). This equation will contain but one arbitrary constant; since the condition to pass through a given point will determine the curve to be one of a limited number of curves of the system, this number being indicated by the degree of equation (1) with respect to \( p \). The equation of the complete primitive will therefore be of the form,
   \[ f(x, y, c) = 0; \] \[ (3) \]
and as the number of curves of the system (3) passing through a given point is indicated by the degree of the equation with respect to \( c \), it is evident that \( p \) and \( c \) will occur in equations (1) and (3) respectively in the same degree.

3. Let us suppose that the system of curves represented by the complete primitive admits of an envelop: then a point moving in the envelop will always have the same direction as if it were moving in that one of the system of which it constitutes the point of contact with the envelop. Hence a point so moving will satisfy the differential equation; in other words the equation of the envelop or any branch of it will form a solution of eq. (1). Such a solution contains no arbitrary constant, and is called a Singular Solution.

4. If the differential equation is of the second degree with respect to \( p \) it may be written in the form

\[
Ap^2 + Bp + C = 0, \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (4)
\]

\( A, B \) and \( C \) denoting functions of \( x \) and \( y \). In this case there will generally exist a certain region of the plane for which \( p \) is impossible, and a certain region for which \( p \) has two real and different values. While \( c \) passes through a complete cycle of values, the curve (3) sweeps twice over each point in the latter region, and the boundary between the two regions, where the values of \( p \) are equal, is the envelop of the system, that is, the curve whose equation is the singular solution. To find this solution we have then only to form the equation

\[
B^2 - 4AC = 0, \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (5)
\]

the condition for equal roots in (4). Thus the differential equation being

\[
xp^2 - py^2 + a = 0
\]

the singular solution is

\[
y^2 = 4ax.
\]

5. It is to be observed however that we may in this manner obtain an equation which does not satisfy the given differential equation. The above reasoning, in fact, while it shows that the locus of (5) includes the envelop if there be one, it does not show that it can include no other branches. In the first place even when the branch in question is the boundary between the two regions mentioned in Art. 4, it may be the locus of a cusp in the complete primitive, and a point moving in such a locus will not generally satisfy the differential equation. In the second place the branch in question may not even form a portion of the boundary; for it may be the locus of the point of contact of two curves belonging to the system (3). As an example of the first case let the given equation be

\[
ap^2 - py + x = 0.
\]

Eq. (5) gives

\[
y^2 - 4ax = 0,
\]
but it will be found on trial that this is not a solution of the given equation. The expression \( y^2 - 4ax \) being negative within the parabola \( y^2 = 4ax \) and positive outside of it, we conclude that it forms the boundary, and must therefore be the locus of a cusp of the complete primitive. As an example of the second case, given the equation

\[
y^2p^2 + y^2 - a^2 = 0.
\]

Eq. (5) gives

\[
y^2(y^2 - a^2) = 0,
\]

which includes the loci \( y = 0 \) and \( y = \pm a \). The latter satisfy the given equation, but \( y = 0 \) does not. The complete primitive in this case is

\[
y^2 + (x - c)^2 = a^2,
\]

representing (since \( c \) is arbitrary) a circle whose radius is \( a \), and whose centre moves on the axis of \( x \); \( y = \pm a \) constitute the envelop, and \( y = 0 \) is the locus of the point of contact of two circles of the system.

6. The boundary line will necessarily be a part of the locus of eq’n (5), but it may happen that it is also one of the particular integrals included in the complete primitive. The following example from Boole’s Differential Equations illustrates this, though the author employs a different criterion for the singular solutions. Given

\[
p - pxy + y^2 \log y = 0.
\]

Eq. (5) gives

\[
y^2(x^2 - 4 \log y) = 0;
\]

the roots \( y = 0 \) and \( x^2 = 4 \log y \) or \( y = e^{2x} \) are both solutions of the given equation, and \( p \) is only possible in the region included between the curve \( y = e^{2x} \) and the axis of \( x \). The complete primitive in this case is

\[
y = e^x e^{-c^2}, \quad y = e^{2x} \text{ is a proper envelop, but the axis, } y = 0, \text{ is not an envelop, but the particular integral corresponding to } c = \infty.
\]

7. It will be observed that in this example the expr. \( \sqrt{y^2(x^2 - 4 \log y)} \) becomes imaginary when \( y \) passes through zero not by reason of the factor \( y^2 \) but because \( \log y \) becomes imaginary when \( y \) becomes negative. We need only in fact observe the existence of the function \( \log y \) in the differential equation to infer that \( y = 0 \) is a portion of the boundary line. So in the example

\[
p = \frac{y \log y}{x}
\]

we see at once that \( y = 0 \) is the boundary line, for \( p \) is imaginary when \( y \) is negative and real when \( y \) is positive. The complete primitive in this case is \( y = e^x \). When \( c = \infty \) the particular integral gives \( y = 0 \) if \( x \) is negative, that is coincides with one half of the axis of \( x \), while if \( c = -\infty \) the particular integral coincides with the other half of the axis. Thus the boundary is made up of portions of two of the particular integrals.
8. If the equation is of the first degree with respect to $p$, and algebraic with respect to $x$ and $y$, $p$ will be possible for every point of the plane and there will be no singular solution. If the equation is algebraic and of a degree higher than the second with respect to $p$ we may apply the usual condition for equal roots that is to say, the equations $\varphi(x, y, p) = 0$ and $\varphi'(x, y, p) = 0$, where $\varphi' = \frac{d\varphi}{dp}$ must be satisfied by a common value of $p$: hence eliminating $p$ between these equations we have the condition expressed as a relation between $x$ and $y$. For example, given

$$\varphi = p^3 - 4xyz + 8y^2 = 0,$$

then

$$\varphi' = 3p^2 - 4xy = 0;$$

eliminating $p$ we find $y = 0$, and $27y = 4x^2$. Each of these is a branch of the envelop of the complete primitive, and is a singular solution. The complete primitive in fact is $y = c(x - e)^2$, representing a series of parabolas which touch the axis of $x$ and the cubical parabola $27y = 4x^2$.

---

**SOME TRIGONOMETRIC SERIES.**

---

BY PROF. D. TROWBRIDGE, WATERBURGH, N. Y.

1. **Take** the equations

$$\log(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4, \ldots \quad (1)$$

$$\log(1 - x) = -x + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4, \ldots \quad (2)$$

$\theta/1 = \varphi$, $2\cos \theta = e^\varphi + e^{-\varphi}$, $2\sin \theta = e^\varphi - e^{-\varphi}$, \ldots \ldots \ldots (3)

$$\log 2\cos \theta = \varphi + \log(1 + e^{-2\varphi}) = -\varphi + \log(1 + e^{2\varphi}), \ldots \ldots \ldots \quad (4)$$

$$\log 2\sin \theta = \log(-1 + \varphi + \log(1 - e^{-2\varphi}) = \log(-1 - \varphi + \log(1 - e^{2\varphi})), \ldots \ldots \ldots \quad (5)$$

$$2\log 2\cos \theta = \log(1 + e^{2\varphi}) + \log(1 + e^{-2\varphi}), \ldots \ldots \ldots \quad (6)$$

$$2\log 2\sin \theta = \log(1 - e^{2\varphi}) + \log(1 - e^{-2\varphi}). \ldots \ldots \ldots \quad (7)$$

Developing (6) and (7) by (1) and (2), we have

$$2\log 2\cos \theta = e^{2\varphi} + e^{-2\varphi} - \frac{1}{2}(e^{4\varphi} + e^{-4\varphi}) + \frac{1}{3}(e^{6\varphi} + e^{-6\varphi}), \ldots \ldots \ldots \quad (8)$$

$$2\log 2\sin \theta = -(e^{2\varphi} + e^{-2\varphi}) - \frac{1}{2}(e^{4\varphi} + e^{-4\varphi}) - \frac{1}{3}(e^{6\varphi} + e^{-6\varphi}), \ldots \ldots \ldots \quad (9)$$

Whence we have

$$\log 2\cos \theta = \cos 2\theta - \frac{1}{2}\cos 4\theta + \frac{1}{3}\cos 6\theta - \ldots \ldots \ldots \quad (10)$$

$$\log(2\sin\theta)^{-1} = \cos 2\theta + \frac{1}{2}\cos 4\theta + \frac{1}{3}\cos 6\theta + \ldots \ldots \ldots \quad (11)$$

By differentiating (10) we have

$$\tan \theta = 2[\sin 2\theta - \sin 4\theta + \sin 6\theta - \ldots \ldots]. \ldots \ldots \quad (12)$$
2. From (4) we have \[ 2 \varphi = 2 \theta / \sqrt{1 - \log(1 + e^{2\varphi}) - \log(1 + e^{-2\varphi})} \]
\[ = (e^{2\varphi} - e^{-2\varphi}) - \frac{1}{2}(e^{4\varphi} - e^{-4\varphi}) + \ldots. \]
Whence we have \[ \theta = \sin 2 \varphi - \frac{1}{2} \sin 4 \varphi + \frac{1}{3} \sin 6 \theta - \ldots. \]
Multiply this by \( d \theta \) and integrate, multiply again and integrate, and continue this process. Denote the arbitrary constants by \( C_2, C_4, C_6, \&c. \), noticing that when the sine occurs in the series, this constant is zero. We shall have, if \( 1.2.3. \ldots n = [n] \),
\[ \frac{\theta^2}{2} + C_2 \]
\[ \frac{\theta^4}{4} + \frac{C_2 \theta^4}{2} + C_4 \]
\[ \frac{\theta^n}{[n]} + \frac{C_2 \theta^n}{[n-2]} + \frac{C_4 \theta^n}{[n-4]} + \ldots C_n = \frac{(-1)^{\frac{n}{4}}}{2^{\frac{n}{4} - 1}} \frac{\cos 2 \theta - \cos 4 \theta}{2^n} + \cos 6 \theta \frac{3^n}{3} \ldots \]
(15)
Now make \( S_n = \frac{1}{2^n - 1} \frac{1}{2} + \frac{1}{3} - \ldots (16), S_n' = \frac{1}{2^n + 1} + \frac{1}{3} - \frac{1}{4} \ldots (17) \)
\( S_n'' = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} \ldots (18), S_n' - S_n = \frac{1}{2^n - 1} \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \ldots = \frac{1}{2^n - 1} S_n; \)
\[ S_n' = S_n \frac{2^n - 1}{2^n - 1} \ldots (19), S_n'' = \frac{1}{2} (S_n' + S_n) = S_n \frac{2^n - 1}{2^n - 2} \ldots (20) \]
Now make \( \theta = \frac{1}{4} \pi = \pi \) in (15), and also make \( \theta = 0 \), and we find
\[ \frac{\theta^2}{[n]} + \frac{C_2 \theta^2}{[n-2]} + \frac{C_4 \theta^2}{[n-4]} + \ldots + C_n = \frac{(-1)^{\frac{n}{4}}}{2^{\frac{n}{4} - 1}} S_n \ldots (21); C_n = \frac{(-1)^{\frac{n}{4}}}{2^{\frac{n}{4} - 1}} S_n \ldots (22) \]
In these equations \( n \) is even. In (22) make \( n = 2, 4, 6, \&c. \), and substitute in (21) and we have
\[ \frac{\theta^n}{[n]} \frac{S_n}{2^n [n-2]} + \frac{S_4}{2^n [n-4]} + \frac{S_6}{2^n [n-6]} + \ldots + \frac{(-1)^{\frac{n}{4}}}{2^{\frac{n}{4} - 1}} S_n = \frac{(-1)^{\frac{n}{4}}}{2^{\frac{n}{4} - 1}} S_n \ldots (23) \]
From the form of (23) we easily see that \( p^s \) is a factor of all its terms; let us therefore make \( \frac{1}{2^n - 1} S_n = p^s B_n \), and we shall have
\[ 1 - n(n-1)B_2 + n(n-1)(n-2)(n-3)B_4 + \ldots + (-1)^{n-1} n! B_n \frac{2^n - 1}{2^n} = 0. \ldots (24) \]
Now let \( n = 2, 4, 6, \&c. \), in succession, and we shall have
\[ B_2 = \frac{1}{6}, B_4 = \frac{1}{30}, B_6 = \frac{1}{420}, \&c. \ldots \ldots \ldots \ldots \ldots (25) \]
We also have \[ S_n = 2^{n-1} p^s B_n = \frac{1}{2^{n+1}} \pi^n B_n, \quad S_n' = \frac{\pi^n B_n}{2^{n+1} - 4} \ldots \ldots \ldots \ldots \ldots (26) \]
Whence we find \( S_2 = \frac{1}{12} \pi^2, \quad S_2' = \frac{1}{6} \pi^2, \quad S_2'' = \frac{1}{3} \pi^2; \quad S_4 = \frac{7}{18} \pi^4, \quad S_4' = \frac{1}{20} \pi^4, \quad S_4'' = \frac{1}{40} \pi^4. \)
We have the following well-known relation:
\[
\log \cos \theta = \log \left(1 - \frac{2^{3} \theta^{3}}{3 \pi^{3}} \right) + \log \left(1 - \frac{2^{4} \theta^{4}}{3 \pi^{4}} \right) + \log \left(1 - \frac{2^{5} \theta^{5}}{3 \pi^{5}} \right) + \ldots (27)
\]
Developing by (2) we have
\[
\log \cos \theta = -\frac{2^{3} \theta^{3}}{3 \pi^{3}} \left(1 + \frac{1}{3^{3}} + \frac{1}{5^{3}} + \frac{1}{7^{3}} + \ldots \right) - \frac{2^{4} \theta^{4}}{3 \pi^{4}} \left(1 + \frac{1}{3^{4}} + \frac{1}{5^{4}} + \frac{1}{7^{4}} + \ldots \right) \ldots \frac{2^{5} \theta^{5}}{3 \pi^{5}} \left(1 + \frac{1}{3^{5}} + \frac{1}{5^{5}} + \frac{1}{7^{5}} + \ldots \right) \ldots = \frac{2^{3} \theta^{3}}{\pi^{3}} S_{3} - \frac{2^{4} \theta^{4}}{\pi^{4}} S_{4} + \frac{2^{5} \theta^{5}}{3 \pi^{5}} S_{5} - \ldots (28)
\]
If we differentiate (28) we shall have
\[
\tan \theta = \frac{2^{3} \theta^{3}}{3 \pi^{3}} S_{3}'' + \frac{2^{4} \theta^{4}}{\pi^{4}} S_{4}'' + \frac{2^{5} \theta^{5}}{3 \pi^{5}} S_{5}'' + \ldots (29)
\]
or
\[
2 \tan \theta = \frac{2^{3}}{2} B_{2} \theta + \frac{2^{4}}{2} B_{4} \theta^{3} + \frac{2^{5}}{2} B_{6} \theta^{5} + \ldots (30)
\]

3. Let us resume Eq. (12) and make
\[
\tan \varphi = 2[\sin 2\varphi - \sin 4\varphi + \sin 6\varphi - \ldots] = A_{1} \varphi + A_{2} \varphi^{3} + A_{4} \varphi^{4} + A_{4} \varphi^{4} \ldots (31)
\]
If we differentiate and make \( \varphi = 0 \) after each differentiation, we shall find
\[
A_{1} = 2[1 - 2 + 3 - 4 + \ldots], \quad A_{2} = 2, \quad A_{4} = 0, \quad \&c.
\]
\[
[3] A_{3} = -2[1 - 2^{3} + 3^{3} - 4^{3} + \ldots],
\]
\[
[5] A_{5} = 2[1 - 2^{5} + 3^{5} - 4^{5} + \ldots], \quad \&c.
\]
Now make \( 1 - 2^{n-1} + 3^{n-1} - 4^{n-1} + \ldots = (-1)^{n+1} \sum (1)^{n-1} \ldots (32) \) then we shall have
\[
\tan \varphi = 2^{1} \sum (1) \varphi + \frac{2^{3} \sum (1)^{3} \varphi^{3}}{[3]} + \frac{2^{5} \sum (1)^{5} \varphi^{5}}{[5]} + \ldots (33)
\]
We must now find the various sums represented by \( \sum \). If we differentiate (14) we have
\[
1 = 2[\cos 2\varphi - \cos 4\varphi + \cos 6\varphi - \ldots]
\]
\[
0 = \sin 2\varphi - 2 \sin 4\varphi + 3 \sin 6\varphi - \ldots
\]
\[
0 = \cos 2\varphi - 2 \cos 4\varphi + 3 \cos 6\varphi - \ldots, \quad \&c.
\]
Make \( \varphi = 0 \) and we see that
\[
0 = 1 - 2^{n} + 3^{n} - 4^{n} + \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (34)
\]
Now let \( u = (1 + e) - 1 = 1 - e^{2} + e^{2} - e^{2} + \ldots \ldots \ldots \ldots \ldots \ldots (35) \)
e\ being the Napierian base of logarithms. If we differentiate (35) and make 
\[
\frac{du}{dx} = u_{1}, \quad \frac{d^{2}u}{dx^{2}} = u_{2}, \quad \frac{d^{3}u}{dx^{3}} = u_{3}, \quad \&c.
\]
and retain the same notation when we make \( x = 0 \), we shall have
\[
u_{1} = - \sum (1) = - \frac{1}{2}, \quad u_{2} = - \sum (1)^{2} = 0, \quad u_{4} = - \sum (1)^{4}, \quad \&c.
\]
From (35) we have \( u e^{x} + u = 1 \). \( \cdots \) \( e^{x}(u_{1} + u_{1}) + u_{1} = 0, \quad e^{x}(u_{1} + 2u_{1} + u_{2} + u_{2}) = 0, \) and generally
\[
e^{x} \left[ u + n u_{1} + \frac{n(n-1)}{2} u_{2} + \frac{n(n-1)(n-2)}{3} u_{3} + \ldots + n u_{n-1} + u_{n} \right] + u_{n} = 0
\]
If we make \( x = 0 \), and remember that \( u_{2}, u_{4}, \&c., = 0 \), we shall have
\[ u + nu + \frac{n(n-1)(n-2)}{3}v + \ldots + 2u = 0, \ldots \] (36)

in which \( n \) is an odd number. Or we may make
\[ \frac{1}{n} = n^2(1) - n(n-1)(n-3)\Sigma (1)^3 + \ldots + (-1)^{k(n-1)}2\Sigma (1)^n \ldots \] (37)

If we now make \( n = 1, 3, 5, \&c., \) in succession, we shall have
\( \Sigma (1) = 3, \quad \Sigma (1)^3 = 1, \quad \Sigma (1)^5 = \frac{7}{5}, \quad \Sigma (1)^7 = \frac{11}{7}, \quad \Sigma (1)^9 = \frac{14}{9}, \quad \Sigma (1)^{11} = \frac{18}{11}, \quad \&c. \ldots \) (38)

These values i (33) give
\[ \tan \phi = \psi + \frac{2}{3}\varphi^3 + \frac{2}{5}\varphi^5 + \ldots \] (39)

By comparing (30) and (33) we see that
\[ \frac{2^{n-1}}{2^{n-1}}B_2 = 2, \quad \frac{2^{n-1}}{2^{n-1}}B_4 = \frac{2^{5}}{3}\Sigma (1)^5, \ldots \] (40)

We readily see that \( u_1 = u^2 - u, \) and hence
\[ u_2 = (2u - 1)u_1 = u(u - 1)(2u - 1) = 2u^3 - 3u^2 + u, \] and generally
\[ u_n = A_{n-1}u^{n-1} - A_{n-1}u + A_{n-1}u^{n-1} - \ldots + (-1)^nu, \ldots \] (41)

\[ u_{n+1} = [(n+1)A_{n+1}u^n - nA_{n+1}u^{n-1} + (n-1)A_{n+1}u^{n-2} - \ldots + (-1)^n]u^3 - u, \ldots \] (42)

By means of (41) and (42) we can easily compute the values of \( u_2, u_3, \) \&c., knowing the value of \( u_1. \) We see that the sum of the coefficients of (42) is 0.

Since \( 2u = 1, \) we may make \( 2u = z = 1, \) and then
\[ 2^{n+1}u_n = A_{n+1}z^{n+1} - 2A_{n-1}z^n + 2A_{n-1}z^{n-1} - \ldots + 2^n(-1)^n, \] (43)

\[ 2^{n+1}u_{n+1} = (n+1)A_{n+1}z^{n+2} - 2[nA_{n+1} + (n+1)A_{n+1}]z^{n+1} + \ldots + 2^{n+1}(-1)^{n+1}z \] (44)

Now let us make \( 2^{n+1}A_{n-1} = V_{n-1} \ldots \) (45), then, if \( 2^{n+1}u_n = z_n, \ldots \) (46) we shall have
\[ z_n = V_{n+1}z^{n+1} - V_{n}z^n + V_{n-1}z^{n-1} - \ldots + 2^n(-1)^n, \ldots \] (47)

\[ z_{n+1} = (n+1)V_{n+1}z^{n+2} - 2[V_{n} + 2(n+1)V_{n+1}]z^{n+1} + \ldots + 2^{n+1}(-1)^{n+1}z \ldots \] (48)

Since \( z = 1, \) \( z_n = V_{n+1}z^n + V_{n-1}z^n - \ldots + 2^n(-1)^n, \ldots \) (49)

\[ z_{n+1} = -(n+1)V_{n+1}z^n + nV_{n} - (n-1)V_{n-1}z^n - \ldots + 2^n(-1)^{n+1} \ldots \] (50)

If we now make \( n = 1, \) we have \( V_2 = 1, \) \( V_1 = 2, \) and
\[ z_3 = 2z - 2z = 1 - 2 = -1, \]
\[ z_2 = 2z^2 - 6z^2 + 4z = -2 + 2 = 0, \]
\[ z_3 = 2z^3 + 28z^3 - 8z = -6 + 12 - 4 = 2, \]
\[ z_4 = 24z^4 + 120z^4 + 200z^4 + 24z^7 + 16z = -24 + 72 - 56 + 8 = 0. \]
These equations are easily formed. We find \( u_1 = \frac{1}{6} z_1 = -\frac{1}{6}, u_2 = \frac{1}{6} z_2 = \frac{1}{6}, \) &c. It will be noticed that \( \gamma(n+1) \) has some power of 2 for a denominator, and the relation of these numbers to the Numbers of Bernoulli, is easily ascertained.

If again we make \( 1.2.3.4\ldots n = [n], \) I have found
\[
z_n = [n] z_n^{n+1} - [n+1] z_n^n + \frac{1}{8} [n+1] (3n-2) z_n^{n-1} - \frac{1}{3} [n+1] (n-1)(n-2) z_n^{n-3} + \frac{1}{3} [n+1] (15n^3 - 105n^2 + 230n - 152) z_n^{n-5} - \ldots
\] (51)
The remaining coefficients I have not yet obtained.

If we take
\[
1 + x + x^2 + x^3 + \ldots + x^n = \frac{x^{n+1} - 1}{x - 1},
\]
and make \( x - 1 = y, \) then \( dx = dy \) and
\[
1 + x + x^2 + \ldots + x^n = \frac{(1+y)^{n+1} - 1}{y} = n + 1 + \frac{n(n+1)}{1.2} y + \frac{n(n+1)(n-1)}{1.2.3} y^2 + \ldots
\] (1)

If we differentiate and make \( \frac{n(n+1)}{1.2} = N_1, \frac{n(n+1)(n-1)}{1.2.3} = N_2, \) &c., we shall have
\[
1 + 2x + 3x^2 + \ldots + nx^{n-1} = N_1 + 2N_2 y + 3N_3 y^2 + \ldots
\] (2)
Multiply by \( x = 1 + y, \) and make \( A_1 = N_1 + 2N_2, A_2 = 2N_2 + 3N_3, \) &c., and we shall have
\[
x + 2x^2 + 3x^3 + \ldots + nx^n = N_1 + A_1 y + A_2 y^2 + \ldots
\] (3)

Differentiate again and we have
\[
1 + 2x + 3x^2 + \ldots + nx^{n-1} = A_1 + 2A_2 y + 3A_3 y^2 + \ldots
\] Again multiply by \( x = 1 + y, \) put \( A_1 + 2A_2 = B_1, 2A_2 + 3A_3 = B_2, 3A_3 + 4A_4 = B_3, \) &c., and continue the process, and then make \( x = 1, \) and \( y = 0 \) in these equations, and we shall have, if
\[
S_1^{(p)} = 1 + 2^p + 3^p + \ldots + n^p,
S_2^{(1)} = N_1,
S_2^{(2)} = A_1 = N_1 + 2N_2,
S_2^{(3)} = B_1 = A_1 + 2A_2,
S_2^{(4)} = C_1 = B_1 + 2B_2,
&c. &c. &c.
A_2 = 2N_2 + 3N_3,
B_2 = 2A_2 + 3A_3,
&c. &c. &c.
A_3 = 3N_3 + 4N_4,
&c. &c. &c.
SOLUTION OF THE GENERAL EQUATION OF THE FIFTH DEGREE.

TRANSLATED BY DR. A. B. NEILSON, DANVILLE, KENTUCKY.

(Continued from page 173, Vol. III.)

§. 11.

The function \( b = (yz + yz + yz + vy + vz + vu + vt + xu + xt + ul) \) is symmetrically halved, if I separate it into two parts (of which each has 5 terms) whose product \( = y^5 v^2 x^2 u^2 t^2 \).

There are 12 such symmetrical halves, viz.:

I.

\[
\begin{align*}
& (yz + yz + vz + xu + ul) \\
& (yz + yu + vz + vt + ul) \\
& (yz + yu + vt + xu + xt) \\
& (yz + yv + vz + xu + xu) \\
& (yz + yz + vu + vt + xu) \\
& (yz + yz + vz + xu + ul)
\end{align*}
\]

and consequently the expression for the same, which is a function of \( b \cdot c \cdot d \cdot e \), must have an irrational part which has 12 harmonic significations.

Of the above symmetrical halves the sum of any two \( = b \), or of any six \( = 3b \), as is signified in I. by their juxtaposition and superposition.

The function \( c = (yz + yu + yz + yz + yu + vz + xu + vt + xu + xu) \) is symmetrically divided if I separate it into two parts (of 5 terms each) whose product \( = y^5 v^2 x^2 u^2 t^2 \).

There are 12 such symmetrical halves, viz.:

II.

\[
\begin{align*}
& (yz + yz + yz + vz + xu + xu) \\
& (yz + yu + yz + vz + xu + xu) \\
& (yz + yu + yz + vz + xu + xu) \\
& (yz + yv + yz + vz + xu + xu) \\
& (yz + yz + yu + vz + xu + xu) \\
& (yz + yz + yu + vz + xu + xu)
\end{align*}
\]

of which the sum of any two \( = c \) and of any six \( = 3c \), as this is signified by their disposition.

There are relations between the symmetrical halves of \( b \) and those of \( c \). I mention one of these. Among the halves of \( b \) and \( c \) there are always two whose terms multiplied singly with one another give the product \( y \cdot v \cdot z \cdot u \cdot t \) five times. Such corresponding halves are:

\[
\begin{align*}
m &= (yz + yz + vz + xu + ul) \quad \text{with} \quad (yz + yv + yu + vz + xu + xu) = p \quad \text{and} \\
\overline{m} &= (yz + yu + yz + vz + vt + ul) \quad \text{with} \quad (yu + yu + yz + vz + vt + ul) = p.
\end{align*}
\]
Hence \( mp_1 = (yzu + yzu + yzt + vzt + vut + ut) = (y^2zu + \ldots + 20t) + (y^2u + y^2z + v^2zt + z^2uy + y^2t + v^2u + v^2y + z^2tu) \). Analogously, \( mp_2 = (y^2zu + \ldots + 20t) + (y^2z + y^2u + v^2u + v^2z + v^2y + z^2tu) \).

I can therefore represent the symmetrical half \( p \) by a function of \( b, c, d, e \) and \( m \), as I can likewise represent by a homogeneous function

\[
\begin{align*}
& (y^2t^2u + y^2z^2 + v^2zt + z^2uy + y^2t + v^2u + v^2y + z^2tu), \\
& (y^2x^2 + y^2u^2 + v^2u^2 + v^2y^2 + z^2tu).
\end{align*}
\]

§. 12.

Let an equation of the 5th degree be constructed with the 5 unknown quantities \( yu, yt, vz, zu, ut \), (the 5 terms of the symmetrical half \( m \)).

If I put: \( (yu + yt + vz + zu + ut) = m, \)
\( (yt + yz + yut + vzu + xu) = p, \)
\( d + (y^2zt + v^2yz + z^2vu + u^2zt + y^2tu) = n, \)
\( (y^2z^2 + y^2u^2 + v^2zu + v^2ut + z^2ty) = q, \)

it follows that the equation to be constructed is:

I. \( x^5 - mx^4 + nx^3 - gx^2 + epx - e^5 = 0. \)

Analogously I construct for the unknown quantities \( yz, yu, vu, vt, xt \) the equation:

II. \( x^5 - m'x^4 + n'x^3 - gp^2 + epx - e^5 = 0, \)

in which \( m_i = b - m, \)
\( n_i = d + (y^2uz + v^2ut + z^2yt + u^2yn + t^2zu), \)
\( q_i = (y^2u^2 + y^2z^2 + v^2u^2 + v^2v^2 + u^2zt + z^2tu), \)
\( p_i = c - p. \)

If I now multiply together equations I. and II. I thus get an equation of the 10th degree, whose unknown quantities are the ten combinations of \( y, v, z, u \) and \( t \) taken 2 at a time. This multiplication gives:

III. (1) \( x^{10} - (m + m) + (n + n, + mm) + (q + q, + nm, + n'm) + \]
\( + [e(p + p, + gm, + q, + mn,)] + (2e^2 + e(pn, + p, + m + q, + qn,)] + \]
\( + [e^2(m + m, + m, + pn, + p, + n, + q,),] + (e^2(n + n, + e(p, + q, + q,)] + \]
\( + [e^2(q + q, + pp,)] + e^3(p + p, + e) = 0. \)

or (2) \( x^{10} - bx^8 + bx^6 - Cx^8 + Dx^6 - Ex^4 + Fx^2 - Gx^2 + Hx^2 - e^2cx + e^3 = 0. \)

The coefficients of equation III. can be represented by functions of \( b, c, d, e. \) Thus:

IV. \( B = - d, \quad F = be^2 - 3ode + d^2, \)
\( C = e^2 - 2bd, \quad G = e^2(b^2 - d), \)
\( D = ce - d^2, \quad H = bde^2, \)
\( E = - 2bce + bd^2 + 2e^3. \)
From III. and IV. there results:

V.  
(1) \( m_i = b - m \),
(2) \( p_i = c - p \),
(3) \( d = (n + n_i + m) \),
(4) \( s^2 - 2bd = (q + q_i + nm_i + n, m) \),
(5) \( -d^2 = (qm_i + q, m + nm_i) \),
(6) \( -2bce + bd^2 = e(pm_i + p, m) + q, n + q_n \),
(7) \( -3cde + d^3 = e(p, n + p, n) + qq_i \),
(8) \( e(b^2 - d) = e(n + n_i) + q, p + q, q_i \),
(9) \( bd = (q + q_i + pp_i) \).

From (3), \( n_i = -(d + n + nm_i) \). From this and (4), (5), (6), (7) and (8):

VI.  
(1) \( (s^2 - 2bd) = (q + q_i + nm_i - nm - dm - m^2 m) \),
(2) \( (-d^2) = (qm_i + q, m - nd - n^2 - nm_i, m) \),
(3) \( (s^2 - 2bce + bd^2) = e(pm_i + p, m) + q, n - qd - qm_i \),
(4) \( (-3cde + d^3) = e(p, n - pd - pn - pm_i - m) + qq_i \),
(5) \( e(b^2) = (pq_i + p, q - em_i) \).

From V. (9) \( q = bd - q - pp_i \). From this and VI.:

VII.  
(1) \( (s^2 - 3bd) = -(pp_i + nm_i - nm - dm - m^2 m_i) \),
(2) \( (-d^2) = (qm_i + bdm - pm_i - mpp_i - nd - n^2 - nm_i) \),
(3) \( (s^2 - 2bce + bd^2) = e(pm_i + p, m) + nbd - 2qm_i - npp_i - qd - qm_i \),
(4) \( (-3cde + d^3) = e(p, n - pd - pn - pm_i - m) + bdp - q^2 - qpp_i \),
(5) \( e(b^2) = (pq + bdp - pq - pp_i - em_i) \).

From VII. (1) and (5): \( n = \frac{s^8 - 3bd + pp_i + dm + m^2 m_i}{m_i - m} \),
\[ q = \frac{e(b^2) + ps^2 - pd - em_i}{p_i - p} \]

Since \( p_i = c - p \) and \( m_i = b - m \), if \( I \) substitute these expressions in equations VII. (2), (3) and (4), three other equations arise which have only the unknown quantities \( p \) and \( m \). From those again an equation for \( m \) may be found, but which can be constructed more easily (§. 16). While now I do not discover \( m \) from these three equations, but, by retaining this element, find \( p \), this must be capable of being done by means of a final equation of the first degree (for \( p \)), for in case \( m \) is known there is only one \( p \) which sustains to it the relation shown by equation I. to exist between them. Consequently it is clear that \( p \) is a function of \( b, c, d, e \) and \( m \), and thus, when \( m \) is known, I know \( p \) together with \( p_i \).

Hereby it is also proved that the expressions §. 11. III. are functions of \( b, c, d, e \) and \( m \).
§ 13.

Let I. \((25e - 50) + 4(y^2t + y^2zu + v^2yz + v^2ut + x^2zt + z^2vu + u^2yv + u^2zt + t^2yu + t^2ox) + 6(y^2z + y^2zt + y^2z + y^2z + v^2zt + v^2z + v^2y + z^2u + z^2tu + u^2v + v^2y) = S.\)

\((y^2v + x^2z + z^2u + u^2t + t^2y) + 2(y^2u^2 + v^2z + x^2y^2 + u^2v^2 + t^2z^2) + 4(y^2z + v^2zu + z^2u + u^2v + t^2v) + 2(y^2v^2 + y^2z + z^2v^2 + u^2v^2 + t^2y^2) + 4(y^2u + v^2t + v^2v + t^2v) + 2(y^2z + y^2zu + z^2u + u^2v + t^2y) = T.\)

\((y^4u + v^4t + z^4v + u^4v + t^4y) + 2(y^4u^2 + v^4z + z^4u + u^4v + t^4v) = U.\)

\((y^4v + z^4v + u^4v + t^4y) + 2(y^4u^2 + v^4z + z^4u + u^4v + t^4v) = V.\)

If now I represent the expression 0 = \((y + v + z + u + t)^6,\)

\(f(1) = (y + v' + z' + u'' + t''')^6,\)

etc., by the development of the powers indicated, there results:

II. \((y + v + z + u + t)^6 = 0 = 5(S + T + U + V + W)\)

\((y + v + z + u + t)^6 = F(1) = 5(S + T + U + V + W)\)

\((y + v + z + u + t)^6 = F(2) = 5(S + T + U + V + W)\)

\((y + v + z + u + t)^6 = F(3) = 5(S + T + U + V + W)\)

\((y + v + z + u + t)^6 = F(4) = 5(S + T + U + V + W)\)

III. \(F(1) + F(2) + F(3) + F(4) = 25S.\)

If I multiply the expression:

\((y^2vt + y^2zu + v^2yz + v^2ut + z^2vu + z^2yt + u^2yv + u^2zt + t^2yu + t^2vz)\)

by 0 = \((y + v + z + u + t),\) there results:

\(0 = (y^2vzu \ldots 20 t) + (y^2vt + y^2zu + v^2yz + v^2ut + z^2vu + z^2yt + u^2yv + u^2zt + t^2yu + t^2vz) + (y^2v^2 + y^2zu + v^2yz + v^2ut + u^2yv + u^2zt + t^2yu + t^2vz) + (y^2v^2 + y^2zu + v^2yz + v^2ut + u^2yv + u^2zt + t^2yu + t^2vz) + \ldots\)

\(= \quad 5e + (bc + 5c) - \ldots\)

\(= -bc + (y^2v^2 + y^2zu + v^2yz + v^2ut + u^2yv + u^2zt + t^2yu + t^2vz)\)

Whence:

III. \((y^2vt + y^2zu + v^2yz + v^2ut + z^2vu + z^2yt + u^2yv + u^2zt + t^2yu + t^2vz)\)

\(= -bc + (y^2v^2 + y^2zu + v^2yz + v^2ut + u^2yv + u^2zt + t^2yu + t^2vz + \ldots)\)

*For want of sorts Roman letters are here used.—Compositor.
If the result of this equation is introduced into the expression for \( S \) (I. preceding), there follows:

\[
IV. \quad S = 5[(5e - bc) + 2(y^5v^2u + y^2z^2t + z^3u^2y + u^2tv + y^2z^2v \\
+ y^2u^2t + v^2u^2z + z^2u^2t + u^2tv) + 25e],
\]

which I write \( S = 5s \).

From this and III.:

\[
V. \quad F(1) + F(2) + F(3) + F(4) = 125s,
\]

and it is shown, by the relations in §. 11. III. and §. 12. VII., that \( s \) is a function of \( b, c, d_e, e \) and \( m \).

\[\text{§. 14.}\]

From §. 7. VII. I take:

\[
I. \quad (1) \quad e = \frac{1}{25}[(CE^3 + D^2E)F^3 + (C^3 + D)F],
\]

\[
(2) \quad 125bc + 3125c = (E^2F^3 + D^2F^3 + C^2E^3 + F) + 10(C^3 + DE^3)
+ D^3E^3)F^3 + 10(C^3E + CD^3)F^3.
\]

But now since (§. 7 and §. 13)

\[
(E^2F^3 + D^2F^3 + C^2E^3 + F) = (y - v^2 - z^2 - u^2 + t^2)^2 + (y - v^2 - z^2 - u^2 + t^2)^2
+ (y - v^2 - z^2 - u^2 + t^2)^2 + (y - v^2 - z^2 - u^2 + t^2)^2
= F(1) + F(2) + F(3) + F(4)
= 125s,
\]

I write equation (2) thus:

\[
(3) \quad 25(25e - bc - s) = 2(C^3 + DE^3 + D^3E^3)F^3 + 2(C^3E + CD^3)F^3.
\]

From §. 10. VI. I know:

\[
II. \quad f(1)(4) = \frac{1}{5}[-5b + (2m - b)] \quad \text{and} \quad f(2)(3) = \frac{1}{5}[-5b - (2m - b)] \quad \text{or} \quad p = pq.
\]

The expression \( f(1)(2)(3)(4) = 5(b^2 - m^2 + mb) = pq \).

The expression \( f(1)(2)(3)(4) \) is equated with \( CDEF^3 \), I have

\[\text{III. CDF = q or C = q + DF, } \quad \text{EF = p or E = p + F.}\]

Having substituted both in equation I. there follows:

\[
IV. \quad (1) \quad D^3pF^2 + D^3pF^2 = 25D^3pF^2 + D^3pF^2 + q^3 = 0,
\]

\[
(2) \quad D^3pF^2 + D^3qF^2 + \frac{1}{2}[25(s + 5e - 25e)]D^2F + D^2pF^2 + q^2p = 0.
\]

Whence, first, from:

\[
(D^3pF^2 + D^3qF^2 + \frac{1}{2}[25(s + be - 25e)]D^2F + D^2qF^2 + q^2p) = 0
\]

\[
-(D^3pF^2 + D^3qF^2 - \frac{1}{2}[25(s + be + 2cp - 25e)]D^2F + D^2qF^2 + q^2p) = 0,
\]

there results:

\[
(q - p)D^3F^2 + \frac{1}{2}[25(s + be + 2cp - 25e)]D^2F + D^2qF^2(q - p) = 0,
\]

\[\text{V. (1) } D^2F^2 + 25 \left( \frac{s + be + 2cp - 25e}{2(q - p)} \right)DF + p^2q = 0.
\]

I put:

\[
25 \left( \frac{s + be + 2cp - 25e}{2(q - p)} \right) = 2r,
\]

thus

\[
(2) \quad D^2F^2 + 2rDF + p^2q = 0, \quad (3) \quad DF = -r + \sqrt{r^2 - p^2q}.
\]
Second, from IV, (1) and (2) there results:
\[
(D^p F^2 + D^q F^2 + \frac{1}{2} [25(s + b + 2c - 25e)] D^2 F + Dp^2 q^2 + q^2 p) = 0
\]
\[-(D^p q F^2 + D^q F^2) - 25 q = D^2 F^2 Dp^2 q^2 + q^2 p = 0
\]
\[= D^2 F^2 p(p - q) + \frac{1}{2} [25(s + b + 2c - 25e)] D^2 F + q^2 (p - q) = 0,
\]
VI. (1) or \[D^2 F^2 + 25 \left( \frac{s + b + 2c - 25e}{2p(p - q)} \right) D^2 F + \frac{q^2}{p} = 0.\]
I put
\[25 \left( \frac{s + b + 2c - 25e}{2p(p - q)} \right) = 2w.
\]
thus \(D^2 F^2 + 2w D^2 F + q^2 + p = 0\), (3) \(D^2 F = -w + v(w^2 - q^2 + p)\).
From this and \(V\):
VII. (1) \[D = \frac{D^2 F}{DF} = \left( \frac{-w + v(w^2 - q^2 + p)}{r + v(r^2 - p^2 q)} \right) \]
\[= \frac{-wp + v(w^2 p^2 - q^2 p)}{p^2 q} \]
\[= \frac{-wp + v(w^2 p^2 - q^2 p)}{r + v(r^2 - p^2 q)}\]
(2) \[C = \frac{q}{DF} = \frac{-wp + v(w^2 p^2 - q^2 p)}{p^2 q} = \frac{r + v(r^2 - p^2 q)}{r + v(r^2 - p^2 q)}\]
(3) \[CD = \frac{-wp + v(w^2 p^2 - q^2 p)}{p^2 q} \]
\[= \frac{-wp + v(w^2 p^2 - q^2 p)}{r + v(r^2 - p^2 q)}\]
(4) \[F = \frac{q}{CD} = \frac{p^2 q^2}{-wp + v(w^2 p^2 - q^2 p)} \]
\[= \frac{r + v(r^2 - p^2 q)}{r + v(r^2 - p^2 q)}\]
(5) \[E = \frac{p}{F} = \frac{-q^2 p}{wp + v(w^2 p^2 - q^2 p)} \]
\[= \frac{-wp + v(w^2 p^2 - q^2 p)}{r + v(r^2 - p^2 q)}\]
From (4) it now follows immediately, since \(F(1) = F, F(2) = C F^2, F(3) = D F^3, F(4) = E F^4\),

VIII. \(f(1) = \sqrt{\left( \frac{-wp + v(w^2 p^2 - q^2 p)}{q^2} \right) \left( r + v(r^2 - p^2 q) \right)^2},\)
\(f(2) = \sqrt{\left( \frac{-wp + v(w^2 p^2 - q^2 p)}{p^2} \right) \left( r + v(r^2 - p^2 q) \right)^2},\)
\(f(3) = \sqrt{\left( \frac{-wp + v(w^2 p^2 - q^2 p)}{p^2} \right) \left( r - v(r^2 - p^2 q) \right)^2},\)
\(f(4) = \sqrt{\left( \frac{-wp - v(w^2 p^2 - q^2 p)}{p^2} \right) \left( r + v(r^2 - p^2 q) \right)^2}.\)
So that in the construction of the formula for \(y\), the choice is between: IX. (1) \(y = \frac{1}{4} [f + C f^2 + D f^3 + E f^4]\)
and \(y = \frac{1}{4} [v F(1) + \frac{1}{4} F(2) + \frac{1}{4} F(3) + \frac{1}{4} F(4)].\)

For use the first form, by which the mutual fitness of the 4 fifth roots is best represented, commends itself.

(To be concluded in No. 2.)
SOLUTION OF THE SIX ORIGINAL PROBLEMS PUBLISHED IN NO. SIX, VOL. III.

BY THE PROPOSER.

Solution of I. Let $d, d'$ be the sun's distance from any two circular planetary orbits; $v, v'$, the orbital velocities of the two planets; $m, m'$, their masses; $g, g'$, their intensities of gravity towards the sun; let $i, i'$ be those parts of gravity which are converted into the orbital forces; $s$, the orbital distance over which a planet moves, while gravity is being transmitted from the sun to the planet; and let $F, F'$ be the required orbital forces and $R, R'$, the required resisting forces.

With the distances $d$ and $s$ form a right-angled parallelogram, the diagonal of which will represent the compound force of gravity; this force may be resolved into two constituent forces; one called the central force; the other, the orbital force: the central force varies as the intensity of gravity; the orbital force varies as its intensity multiplied into the abberating velocity. Also in the planetary system, the diagonal and $d$ are so very nearly equal, that the latter can be used for the form without any appreciable error.

When the abberating velocity is constant, the orbital force must vary as the force of gravity; therefore

$$i : i' :: g : g'.$$

Newton's law gives

$$g : g' :: m + d^2 : m' + d'^2;$$

hence

$$i : i' :: m + d^2 : m' + d'^2. \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1)$$

But in circular orbits, the abberating velocities are equal to the planetary velocities; also by the well known law of planetary velocity, we have

$$v : v' :: 1 + \sqrt{d} : 1 + \sqrt{d'};$$

multiply by (1)

$$v : v' :: 1 + \sqrt{d} : 1 + \sqrt{d'};$$

But

$$F : F' :: vi : v'i';$$

therefore

$$F : F' :: m + \sqrt{d} : m' + \sqrt{d'} :: R : R'.$$

In the problem, it is assumed that the permanent form of the orbits is circular; hence the resistances in any two circular orbits must be, not only proportional, but exactly equal to the propelling orbital forces; hence $R = F$ and $R' = F'$.

Solution of II. Let $D, D'$ be the densities of the ethereal medium, at the respective distances of any two planets from the sun.

As the resistance depends upon the mass, square of the velocity, and density, we have
\[ R : R' :: m\omega^2 D : m'\omega'^2 D'. \]

But \[ \omega^2 : \omega'^2 :: 1 + \Delta : 1 + \Delta'; \]

hence \[ m\omega^2 D : m'\omega'^2 D' :: mD + \Delta : m'D' + \Delta'; \]

therefore \[ R : R' :: mD + \Delta : m'D' + \Delta'. \]

By I. we have \[ R : R' :: \frac{m - v}{d} : \frac{m' - v}{d'}; \]

therefore \[ D : D' :: 1 + \frac{v}{d^2} : 1 + \frac{v}{d'^2}. \]

Therefore the density of the ethereal medium must vary inversely as the cube of the square root of the sun's distance, in order to render the two antagonistic forces in circular orbits, equal.

**Solution of III.** First, find the intensity of the earth's gravity towards the sun, compared with the intensity of gravity towards the earth's center, which may be assumed equal to unity. We have

\[
\frac{91430000}{3955.94943182} = 23364.80826992676 = d.
\]

We also have

\[
\frac{\text{sun's mass}}{d^2} = \frac{314760}{d^2} = g
\]

\[ = .000576574051819522; \text{that is, the gravity towards the earth's center is to the gravity of the earth towards the sun as } 1 : G.\]

Second, find the intensity of the accelerating orbital force, exerted upon the earth. The earth moves in its orbit 8976.11926 miles, while gravity is transmitted over the distance \(d\); hence we have

\[ 8976.11926 : 91430000 :: 1 : 10185.91635788, \]

or

\[
\frac{\text{orbital force}}{\text{central force}} = \frac{1}{10185.91635788};
\]

that is, the earth is accelerated in its orbit, by the \(10185.91635788^{th}\) part of its central or gravitating force toward the sun.

This intensity, by the following proportion, can be expressed in terms of terrestrial gravity.

\[ 10185.91635788 : 1 :: .000576574051819522 \]

\[ : .000000056605025170217 = G_1. \]

\(G_1\) is the intensity of the earth's accelerating orbital force. (the intensity of gravity towards the earth's center being unity.)

Third, find the weight of the whole earth = 13, 502, 160, 021, 599, 966, 659, 335, 933 lb. This multiplied by \(G_1\) gives

\[ 764, 290, 107, 874, 962, 825 \text{ lb} = p. \]

This is the pressure in pounds weight in the direction of the earth's orbital motion; and is also the resistance in pounds weight necessary to maintain the earth in a circular orbit.

**Solution of IV.** In an elliptic orbit, the angular velocity of a body around the lower focus, varies inversely as the square of the distance from the focus. (See Dr. Stewart's First Math. Tract, Prop. VI.)
The real velocity in an ellipse may be resolved into two velocities; one in the direction of the radius vector, and the other in a direction perpendicular to the radius vector; the former has no aberrating effect; the latter gives rise to the aberrating velocity of gravity.

The aberrating velocity varies inversely as the distance of the moving body from the focal force.

Let \( d \) and \( d' \) be the distances from the focal force of any two points in an ellipse; let \( V, V' \) be the angular velocities of a body at those two points, and let \( v \) and \( v' \) be the aberrating velocities, or those parts of the velocities which are at right angles to the radius vector.

The actual velocities at right angles to the radius vector must be as the angular velocities multiplied into the respective distances: hence
\[
\frac{v}{v'} = \frac{Vd}{V'd'}.
\]

But
\[
\frac{V}{V'} = \frac{1+\alpha^2}{1+\alpha'^2};
\]

hence
\[
\frac{Vd}{V'd'} = \frac{1+\alpha}{1+\alpha'};
\]

therefore
\[
\frac{v}{v'} = \frac{1+\alpha}{1+\alpha'}.
\] (1)

But the intensity of the aberrating force varies inversely as the square of the distance; hence
\[
\frac{i}{i'} = \frac{1+\alpha^2}{1+\alpha'^2};
\]

multiply by (1) and we have
\[
\frac{iv}{i'v'} = \frac{1+\alpha^2}{1+\alpha'^2};
\]

by prob. I.
\[
\frac{F}{F'} = \frac{iv}{i'v'};
\]

therefore
\[
\frac{F}{F'} = \frac{1+\alpha^2}{1+\alpha'^2}.
\]

Thus it is proved that the aberrating or orbital force, in an elliptic orbit, varies inversely as the cube of the distance from the focal force.

SOLUTION OF v. As the square of the velocity enters as a function in determining the resistance, it will be necessary to show how this function is obtained.

Let \( d, d' \) be any two distances of a body from the lower focus, \( k, k' \), the distances from the upper focus; \( h, h' \), the distances of the apsidal points from the lower focus; \( p, p' \), the perpendiculai points let fall from the lower focus on the two tangents drawn from the points in the orbit at the distances \( d \) and \( d' \), and \( v \) and \( v' \), the elliptic velocities of the moving body at the two points.

From the first Math. Tract of Dr. Matthew Stewart, Prop. 21, Cor., we have
\[
p^2 : h \times h' = d : k;
\]

hence
\[
p^2 = (h \times h' \times d) \div k.
\]

Also
\[
p'^2 : h \times h' = d' : k';
\]

hence
\[
p'^2 = (h \times h' \times d' \div k').
\]

Therefore
\[
\frac{p^2}{p'^2} \cdot \frac{h \times h' \times d}{k} = \frac{h \times h' \times d'}{k'} \cdot \frac{d}{k} = \frac{d'}{k'}.
\]


hence \( \frac{1}{p^4} : \frac{1}{p'^4} :: \frac{k}{d} : \frac{k'}{d'} \)

but (see James Adams' Ellipse, centripetal forces, Prop. I., Cor. I.)

\[ v^2 : v'^2 :: 1 + p^2 : 1 + p'^2; \]

therefore \( v^2 : v'^2 :: k + d : k' + d' \).

In determining the resistance, the density must be multiplied into the square of the velocity.

Let \( D, D' \) be the densities of the resisting medium, at the two points in the ellipse, and let \( R \) and \( R' \) be the respective resistances at those points.

By II. we have \( D : D' :: 1 + \sqrt{d^3} : 1 + \sqrt{d'^3}; \)

multiply by (1) \( v^2 D : v'^2 D' :: k + \sqrt{d^5} : k' + \sqrt{d'^5}. \)

But \( R : R' :: v^2 D : v'^2 D' \);

therefore \( R : R' :: k + \sqrt{d^5} : k' + \sqrt{d'^5}. \)

In problems IV. and V., the mass is not taken into the calculation, because the mass in any one elliptic orbit, is considered constant, and therefore, cannot affect either of the variable forces.

**Solution of VI.** Let \( v', V, g', r' \) be respectively, the real velocity, the aberrating velocity, the intensity of the orbital force of gravity and the resistance, at the mean distance of a body in any elliptic orbit around the sun: let \( m \) = the mass of the earth and \( m' \) = the mass of the planet or comet revolving in the ellipse; let \( V', g', \) and \( r'' \) be the aberrating velocity, the intensity of the orbital force of gravity, and the resistance, at the point in the ellipse where the two opposite forces are equal, and let \( x \) = the required length of the radius vector to the point of equal forces.

\( v' \) can be found from the proportion

\[ 1 + \sqrt{a} : 1 + \sqrt{a'} :: v : v'; \]

hence \( v' = v/\sqrt{a + \sqrt{a'}}. \)

In an elliptic orbit, at the mean distance, the aberrating velocity \( V = \text{the angular velocity}; \)

hence we have

\[ a' : b' :: v' : V :: v/\sqrt{a + \sqrt{a'}} : V; \]

hence \( V = b'v/\sqrt{a + \sqrt{a'}}. \)

By Newton's law of gravity, we have

\[ m + a^2 : m' + a'^2 :: g : g'; \]

hence \( g' = m'g/\sqrt{a + \sqrt{a'}}. \)

Therefore \( Vg' = b'vgm/\sqrt{a + \sqrt{a'}}. \)

But, by problem IV., it is proved that

\[ Vg' : Vg'' :: \frac{1}{a'^3} : \frac{1}{x^3}; \]

hence \( \frac{V'g''}{m/\sqrt{a'^3}} :: \frac{1}{a'^3} : \frac{1}{x^3}; \)

therefore \( V'g'' = \frac{V'g'm/\sqrt{a'^3}}{x^3} = \frac{b'vgm/\sqrt{a^5}}{m/\sqrt{a'^3}x^3} \)

equal the accelerating force at the distance \( x. \)
We also have \( m + \sqrt{a^5} \) : \( m' + \sqrt{a'5} \) :: \( r : r' \); \( \therefore r' = rm'\sqrt{a^5} + m\sqrt{a^5} \).

And by problem V., we have

\[
\frac{r'}{r'} :: \frac{k}{V a^5} : \frac{k'}{V x^5} :: \frac{a'}{V a^5} : \frac{2a' - x}{\sqrt{a^5}};
\]

or

\[
\frac{r'm'\sqrt{a^5}}{m\sqrt{a^5}} : \frac{r''}{r'} :: 1 : \frac{2a' - x}{\sqrt{x^5}};
\]

therefore

\[
r'' = \frac{(2a' - x)r'm'\sqrt{a^5}}{a'm\sqrt{x^5}}
\]

= the resisting force at the distance \( x \).

But at the distance \( x \), the accelerating and resisting forces are (by hypothesis) equal. Therefore we have

\[
\frac{b'vgm'\sqrt{a^5}}{m\sqrt{a'x^5}} = \frac{(2a' - x)r'm'\sqrt{a^5}}{a'm\sqrt{x^5}}.
\]

This reduced becomes

\[
x^3 - 4a'x^2 + 4a''x - v^2g^2b'2a' + r^2 = 0.
\]

This equation, numerically solved, gives one of the values of \( x \) equal to the required radius vector. When the two forces are assumed equal in circular orbits, \( x \) never becomes less than the mean distance, nor greater than the aphelion distance. It is evident that there are two radii vectores of the same value, where the moving body is acted upon by two equal and opposite forces; from these two points, by the way of the aphelion, the aberrating force preponderates; and by the way of the perihelion, the resisting force is in excess. The sum of the resisting forces, in an elliptic orbit, is always greater than the sum of the aberrating forces. The excess of the former diminishes as the eccentricity diminishes; and when the orbit is reduced to a circle, the two sums become equal. By reason of such excess of resistance, hyperbolic and parabolic orbits will be reduced, first, to ellipses of great eccentricity; and second, by the same cause, the eccentricities and periodic times will continually diminish until the orbits become permanent and the sums of the two forces are equally balanced.

[If the foregoing propositions are presented as the basis of a theory of gravitation, it may be objected that the theory presents a phenomenon without a conceivable cause; viz., an elastic ether with a density increasing toward the centers of bodies immersed in it.

If gravitation is the result of ethereal pressure, the only conceivable mode of its manifestation is through a density decreasing toward the center.

Experiment proves that increased local vibration in an ether, or gas, is attended by diminished density at that point, and consequent motion toward the point of increased vibratory motion. Hence it would seem that the conclusion arrived at, that the density of the ether decreases as a function of the distance from the center of force increases, disproves the hypothesis.—Ed.]
FORCE OF GRAVITY AT ANY LATITUDE.

BY R. J. ADCOCK, MONMOUTH, ILL.

On the supposition that the earth's surface is an oblate ellipsoid of revolution, and a level surface, to determine the formula for the superficial force.

Let \( a \) = the equatorial radius, \( b \) = polar radius and \( r \) = radius at latitude \( \ell \); put \( v \) = geocentric latitude; \( p \) = equatorial, and \( q \) = polar gravity, and \( R \) = gravity at the latitude \( \ell \). From the geometrical relations,

\[
r = \frac{ab}{(a^2 \sin^2 v + b^2 \cos^2 v)^{\frac{1}{2}}} \quad \ldots \ldots \ldots \ldots (1)
\]

\[
dr = \frac{a^2 \sin^2 v db + b^2 \cos^2 v da}{(a^2 \sin^2 v + b^2 \cos^2 v)^{\frac{3}{2}}} \left[ \frac{(b - a)\tan^2 l + db + da}{[1+(b^2 + a^2)\tan^2 l]^{\frac{1}{2}}} \right] \quad \ldots \ldots (2)
\]

\[
\cos(\ell - v) = \frac{\cos \ell [1+(b^2 + a^2)\tan^2 l]}{[1+(b^2 + a^2)\tan^2 l]^{\frac{1}{2}}} \quad \ldots \ldots \ldots (3)
\]

From the principles of fluid equilibrium, \( R \) is everywhere perpendicular to the level surface, and is inversely proportional to the distance between the two contiguous level surfaces at \( \ell \). And it is evident that since the force \( R \) makes the exterior level surface an ellipsoid, the force \( R = dR \), which makes the surface infinitely near this level, makes it an ellipsoid also.

Hence

\[
R \cos (\ell - v) dr = pda = qdb. \quad \ldots \ldots \ldots (4)
\]

By elimination

\[
R = \frac{p[1+(b^2 + a^2)\tan^2 l]^{\frac{1}{2}}}{\cos \ell [1+(b^2 + a^2)\tan^2 l]^{\frac{1}{2}}}, \quad \ldots \ldots \ldots (5)
\]

From (5) three pendulum observations determine \( p, q, \) and \( b + a \). And the formula agrees tolerably well with the observations given on page 96 of Baily's report of Foster's experiments.

Clairaut's formula which is \( R = p \left\{ 1 + [(q - p) + p] \sin^2 \ell \right\} \) gives an excess of about \( 3 \frac{1}{4} \) vibrations of the seconds pendulum in 24 hours at latitude \( 45^\circ \).

SOLUTION OF A PROBLEM.

BY W. E. K. GOUDY, A. M. SPRINGFIELD, OHIO.

Problem. I have 4 round balls, each 6 inches in diameter, and desire to have a box made of 4 equal boards each one inch thick, that will just circumscribe the balls when they touch each other. Required the square feet of boards in the box?
SOLUTION. Put \(2a = \text{the linear edge, } b = \text{the whole superficials and } c = \text{the solid contents of the tetrahedron formed by the four boards; and let } r \text{ and } R \text{ represent the radii of its inscribed and circumscribed spheres.}

Then \(2a = 2r\sqrt{6}; b = 24r^2\sqrt{3}; c = 8r^3\sqrt{3}, \text{ and } 3r = R. \text{ For, let } ABC \text{ represent the base of the tetrahedron and draw the perpendiculars } AN \text{ and } CP \text{ bisecting } AB \text{ and } BC \text{ in } P \text{ and } N. \text{ At the intersection of } AN \text{ and } CP \text{ make } ED \text{ perpendicular to } AN, \text{ and suppose it also perpendicular to the plane of the paper. Make } AD = AC; \text{ then } ED \text{ will be the altitude of the tetrahedron, and } ADN, \text{ a vertical section through one of its edges and its axis. Make } NM = NE, \text{ and draw } MO \text{ perpendicular to } ND; \text{ then } O \text{ be the common center of } r \text{ and } R. \text{ Because } AC = 2a, \therefore PC = AN = DN = a\sqrt{3}. \text{ PC : PB :: NC : NE = } \frac{a\sqrt{3}}{3} = EP. \therefore AE = \frac{a\sqrt{3}}{3}, \text{ and } DE = \frac{a\sqrt{6}}{6}. \text{ Also, by similar triangles, } DN : NE :: DO (= ED - r) : OM = OE = r = \frac{a\sqrt{6}}{6}. \therefore a = 3\sqrt{6}. \ldots (A)

This shows that } OE = \frac{1}{2}ED \text{ and that } 3r = R.

Again, \(4(AP \times CP) = b = 4a^2\sqrt{3} ; \text{ but } a = r\sqrt{6}, \ldots b = 24r^2\sqrt{3}. \ldots (B)

Lastly, \(\frac{1}{2}(DE \times AP \times CP) = c = \frac{1}{2}(2a^2\sqrt{3}) = 8r^3\sqrt{3}. \ldots \ldots \ldots \ldots (C)

If \(2a' = \text{the linear edge of a tetrahedron whose angles are at the centers of the four balls when in position touching each other; then is } a' = 3 \text{ inches. Hence, as the boards are } 1 \text{ inch thick, } OE = \frac{1}{2}\sqrt{6} + 3 + 1 = r, \text{ and, from } (A), \text{ we have } a = 3 + 4 \sqrt{6}. \text{ And } 2a_1 = \text{the linear edge of the insides of the box, } a_1 = 3 + 3 \sqrt{6}.

From (A) we have \(r = \frac{a\sqrt{6}}{6} = \frac{1}{2}\sqrt{6} + 4. \text{ Substituting this value of } r \text{ in } (B) \text{ we get } b = 7.880242 \text{ square feet. Substituting the two values of } r, \text{ corresponding to } a \text{ and } a_1, \text{ in } (C), \text{ we find for the difference of the two tetrahedrons, or the solidity of the box, } 931.4226928 \text{ cubic inches.}

Because \(2a = 25.5959716 \text{ inches, } \therefore r = \frac{1}{2}\sqrt{6}; R = \frac{1}{2}\sqrt{6}; b = a\sqrt{3}, \text{ and } c = \frac{1}{2}a^2\sqrt{2}.

SOLUTION OF PROBLEMS IN NO. FIVE, VOL. III.

129. "Three lines through the vertices of a triangle meet in a point, } P. \text{ Through the intersection of each with the opposite side a perpendicular to that side is drawn and these three perpendiculars meet in a point. } \text{ Find the locus of } P. "
SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

Let $D$, $E$, $F$ be the points of intersection of the three lines with the sides of the triangle, $O$ the common point of the perpendiculars at $D$, $E$, $F$. Draw $PL$, $PM$, $PN$ perpendicular to $BC$, $CA$, $AB$. Put $PL = a$, $PM = \beta$, $PN = \gamma$.

Then we easily find $BD = \frac{ac\gamma}{aa + b\beta}$,

$DC = \frac{ab\beta}{b\beta + \alpha \beta}$, $CE = \frac{b\alpha a}{\alpha \beta + aa'}$,

$EA = \frac{bc\gamma}{bc + aa'}$, $AF = \frac{ob\beta}{oa + b\beta}$,

$FB = \frac{coa}{aa + b\beta}$; and by trigonometry we have

$OF = AE \csc A = \cot A = BD \csc B = BF \cot B$,

whence $B(DC - BD) + CA(EA - CE) + AB(FB - AF) = 0$.

By substitution we have

$a^2\left(\frac{b\beta - \alpha \beta}{b\beta + \gamma}\right) + b^2\left(\frac{\gamma - aa}{c\gamma + aa}\right) + c^2\left(\frac{aa - b\gamma}{aa + b\beta}\right) = 0$,

whence by reduction we find

$a^2c^2(\beta \cos B - \gamma \cos C) + b^2c^2(\gamma \cos C - a \cos A) + c^2a^2(\beta \cos B - a \cos A) = (1)$

the trilinear equation to the locus of $P$, which is, therefore, a line of the third order.

By tracing the curve we find that it consists of three disconnected branches. From $(1)$ and $aa + b\beta + \gamma = 2\Delta \ldots (2)$, by making $\gamma = 0$, we find $a = 0$, $c \sin B$, or $b \cos A \sin B$, and $\beta = c \sin A$, 0, or $a \sin A \cos B$; hence the three branches pass respectively through $B$, $A$, and $K$, and $AK = a \cos B$.

In a similar manner we find that the branch through $K$ passes through $C$, that the branches through $A$ and $B$ intersect $AC$ and $BC$ in $H$ and $G$, and that $AH = a \cos C$, and $BG = b \cos C$. The branch that passes through the vertex opposite the longest side intersects that side, but the other branches do not intersect the sides opposite the vertices through which they pass.

The values $aa = \frac{1}{2} \Delta$, $b\beta = \frac{1}{2} \Delta$, $c\gamma = \frac{1}{2} \Delta$ satisfy $(1)$; hence the curve passes through the center of gravity of the triangle. The values $a = 2R \alpha \cos C$, $b = 2R \cos A \cos C$, $c = 2R \cos A \cos B$ satisfy $(1)$; hence the curve passes through the intersection of the three perpendiculars.

If $a = b$, $(1)$ becomes $(a - \beta)(a^2c^2 - a(2a^2 - c^2)(\gamma + \beta) + c^2\gamma^2) = 0$, or $a - \beta = 0 \ldots (3)$, and $a^2c^2 - a(2a^2 - c^2)(\gamma + \beta) + c^2\gamma^2 = 0 \ldots (4)$
Equation (3) represents the line which bisects the angle \( \angle CB \), and eq. (4) represents two opposite hyperbolas, passing through \( A \) and \( B \), and having the transverse axis parallel to \( AB \). If \( a = b = c \), (1) becomes \((a - \beta) \times (\beta - \gamma)(\gamma - a) = 0\), which is the equation to the bisectors of the angles of the triangle.

134. "Through a point taken at random in the surface of a circle, two chords are drawn, one at random and the other at right angles to the radius through that point; find the average area of the quadrilateral formed by joining the extremities of the chords."

**Solution by Artemas Martin, Erie, Pa.**

Let \( O \) be the center of the circle, \( P \) the random point in its surface, \( AB \) the random chord and \( CD \) the chord at right angles to the radius through \( P \). Draw \( OE \) perpendicular to \( AB \). Let \( OP = x \), and angle \( APO = \varphi \); then \( \angle APC = \frac{\pi}{2} - \varphi \), \( CD = 2y/(r^2 - x^2) \), \( AB = 2y/(r^2 - x^2 \sin^2 \varphi) \), and the area of the quadrilateral \( ABCD = AB \times CD \times \frac{1}{2} \sin APO = 2y/(r^2 - x^2) \times y/(r^2 - x^2 \sin^2 \varphi) \cos \varphi \).

If \( A \) be the average area required, we have

\[
A = \frac{2 \int_0^r \int_0^{\pi/2} (r^2 - x^2)^{3/2}(r^2 - x^2 \sin^2 \varphi)^{1/2} 2\pi x \, dx \, d\varphi}{\int_0^r \int_0^{\pi/2} 2\pi x \, dx \, d\varphi}
\]

\[
= \frac{8}{\pi r^2} \int_0^r \int_0^{\pi/2} (r^2 - x^2)^{3/2}(r^2 - x^2 \sin^2 \varphi)^{1/2} \cos \varphi \, x \, dx \, d\varphi,
\]

\[
= \frac{4}{\pi r^2} \int_0^r (r^2 - x^2) \, dx + \frac{4}{\pi} \int_0^r (r^2 - x^2)^{1/2} \sin^{-1} \left( \frac{x}{r} \right) \, dx,
\]

\[
= \frac{r^2}{\pi} + \frac{1}{\pi} \left[ 2x(r^2 - x^2)^{1/2} \sin^{-1} \left( \frac{x}{r} \right) \right]^r_x + \frac{r^2}{\pi} \left( \sin^{-1} \left( \frac{x}{r} \right) \right)^2 - x^2,
\]

\[
= \frac{1}{2} \pi r^2.
\]

135. "A point is taken at random in the surface of a given circle, and from it a line equal in length to the radius is drawn, so as to lie wholly in the surface of the circle. Find the chance that the line intersects a given diameter."
Let $ABC$ be the given circle, $O$ its center, $AB$ the given diameter, and $P$ the point taken at random.

Put $OP = x$ and angle $AOP = \theta$. With $P$ as center and radius $OA = 1$, describe the arc $CDEG$ cutting $BA$, or $BA$ produced, in $D$ and $G$.

If we suppose $P$ fixed, and $\theta$ less than $\cos^{-1}\frac{1}{4}x$, $G$ is outside the circle, and the required probability is $DE + CDE = [\theta - \sin^{-1}(x \sin \theta) + \cos^{-1}\frac{1}{4}x] + 2 \cos^{-1}\frac{1}{4}x$. But if $\theta$ is greater than $\cos^{-1}\frac{1}{4}x$ and less than $\frac{2}{11} \pi$, $G$ is within the circle and the required probability is $DG = CDE = \cos^{-1}(x \sin \theta) + \cos^{-1}\frac{1}{4}x$.

The probability that $x$ and $\theta$ will have any particular value less than 1 and $\frac{11}{11} \pi$, respectively, is $\pi dx d\theta \frac{1}{4} \pi$. Hence the required probability is

$$
\frac{2}{\pi} \int_0^1 \int_0^{\sin^{-1}(x \sin \theta) + \cos^{-1}\frac{1}{4}x} x dx d\theta + \frac{4}{\pi} \int_0^1 \int_0^{\cos^{-1}(x \sin \theta) \cos^{-1}\frac{1}{4}x} x dx d\theta
$$

in which $u = \cos^{-1}\frac{1}{4}x$. This integral, I think, can only be found by approximate methods.

The foregoing solution is published instead of the solution by Mr. Seitz, which we promised, in No. 6, Vol. III, to publish in this No. We have made the substitution because the method pursued by Mr. Seitz in his solution was subsequently found to be defective.

It will be seen that, in the above solution, the required probability is obtained by taking the sum of the probabilities for the different points.

In the solution obtained by Mr. Seitz, and in which a finite integration is effected, the sum of the intersecting lines from all the points is taken for the numerator, and the sum of all the lines from all the points, for the denominator of the fraction representing the required probability. If the whole number of lines that can be drawn from each point were equal, this method would be correct, but as that is not the case, the probability of drawing the different lines is not the same, and hence the separate probabilities for the different points must be summed for the required probability.

This peculiarity of the question was not observed in our first examination of the solutions, and was, subsequently, pointed out by Mr. Heaton, to whom the manuscript solutions were submitted for examination and comparison.—Ed.]
136. "Suppose a planked floor with thin visible seams between the planks. Let there be a thin straight rod not so long as the breadth of the planks. This rod, being tossed up at hazard, will either fall quite clear of the seams or will lie across one seam. Prove that in the long run the fraction of the whole number of trials in which a seam is intersected, will be the fraction which twice the length of the rod is to the circumference of the circle having the breadth of the plank for its diameter."

**SOLUTION BY WILLIAM HOOVER, BELLEFONTAINE, OHIO.**

Let $2a = \text{width of planks}$, $2b = \text{length of rod}$, $\phi = \text{the angle the rod makes with a perpendicular to the seams}$, and let $x = \text{the distance of the center of the rod from the nearest seam}$. The rod will cross a seam for all values of $x$ from 0 to $b \cos \phi$; hence the chance of crossing the seam is $4\phi + 2\pi$, and as the chance that the center of the rod will take the particular position at the distance $x$ from the seam is $dx + a$, the required probability will evidently be

\[
\frac{2}{\pi a} \int_{0}^{2\pi} \phi dx = \frac{2b}{\pi a} \int_{0}^{2\pi} \phi \sin \phi d\phi = \frac{2b}{\pi a}.
\]

**NOTE, BY CHRISTINE LADD, UNION SPRINGS, N. Y. — THE relation between the sides and diagonals of the contra parallelogram given in problem 118 is a particular case of the relation between the two values of the side of a triangle when the remaining sides and the angle opposite one of them are given. If $c$ be the side required and $B$ the angle given, we have**

\[
c = a \cos B \pm b \cos a,
\]

\[
oc' = a^2 \cos^2 B - b^2 \cos^2 a = a^2 - b^2.
\]

**SOLUTION OF PROBLEMS IN NO. SIX, VOL. III.**

Solutions of problems in No. 6, Vol. III, have been received as follows: From Marcus Baker, 137; Henry Gunder, 137 and 139; H. Heaton, 139, 140 and answer to query; Artemas Martin, 139 and 140; W. L. Marcy, 137 and 140; Prof. J. Scheffer, 138; E. B. Seitz, 137 and 140; Prof. D. Trowbridge, 140; R. J. Adcock, answer to query.

137. "A point $D$, is given in position between two lines which make a given angle at $A$. Find the position of a given line, $BC$, drawn through $D$, and intersecting the two lines in the points $B$ and $C$."

SOLUTION BY MARCUS BAKER, U. S. COAST SURVEY, WASHINGTON, D. C.

Call the given line $BC$, $l$, the angle made by the lines given in position, $a$, and the angle made by the given line $l$, with one of the lines given in position, $\theta$; then, calling the co-ordinates of the given point $a$ and $b$, we have (see Analyst, Vol. II, p. 192)

$$\frac{l}{b \sin a} = \csc \theta + \frac{a}{b} \csc (a + \theta)$$

as the equation of the line.

For brevity put $b \sin a = m$, $b \cos a = n$ and $\csc \theta = x$; whence, since

$$\csc (a + \theta) = \frac{\csc \theta}{\sin a \sqrt{(\csc \theta - 1) + \cos a}}$$

we have

$$\frac{l}{m} = x + \frac{ax}{m \sqrt{(x^2 - 1) + n}} = \frac{(a + n)x + mx\sqrt{(x^2 - 1)}}{n + m \sqrt{(x^2 - 1)}}$$

and hence

$$(mn - mx)\sqrt{(x^2 - 1)} = (a + n)x - ln.$$

Squaring and developing we get, after reduction,

$$x^4 - 2lx^3 + \left[p - \frac{(a + n)^2}{m^2} - 1\right]x^2 + 2l \left[1 + \frac{(a + n)n}{m^2}\right]x - \frac{p}{m^2}(m^2 + n^2) = 0.$$ 

If the given lines are at right angles to each other, i. e., if $a = 90^\circ$ we have $m = b$ and $n = 0$, whence our equation becomes for rectangular co-ordinates

$$x^4 - 2lx^3 + \left[p - (a^2 + b^2) - 1\right]x^2 + 2lx - p = 0.$$ 

Since the equation is of the 4th degree there are in general four positions of the line which fulfill the conditions of the problem.

138. "In a pentagon, $ABCDE$, the triangles $ABE = a$, $ADE = b$, $CDE = c$, $BCD = d$, $ABC = e$ are known; to find the area $F$ of the pentagon."

SOLUTION BY PROF. J. SCHEFFER., COLLEGE OF ST. JAMES, MD.

Denoting the co-ordinates $Am$ by $x$, $Em$ by $y$, $AB$ by $z$, $Ap$ by $t$, $Op$ by $u$, $An$ by $v$, $Dn$ by $w$, we find the equations:

1. $yz = 2a$
2. $zu = 2e$
3. $xz = 2(F - b - d)$
4. $xv + yw = 2b$
5. $tv - uw = 2(F - b - e)$
6. $ty + xu = 2(F - c - e)$.
Multiplying (1) by (5), we obtain

\( yztw - yzw = 4(Fa - ab - ae) \).

Multiplying (2) by (4), we get

\( xxuw + yxw = 4bc \).

Adding (7) and (8), we get

\( xxuc + yxw = 4(Fa - ab - ae + be) \).

Multiplying (3) by (6), we get

\( xxw + ytw = 4(F - b - d)(F - c - e) \).

Equating the right-hand members of (9) and (10), we have

\[ 4(Fa - ab - ae + be) = 4(F - b - d)(F - c - e) \]

whence

\[ F^2 - (a + b + c + d + e)F + (ab + bc + cd + de + ae) = 0 \]

an equation of the second degree symmetrical in form.

139. "Let \( e \) be the eccentricity, or ratio of polar compression to the equatorial radius of the earth. Find the average length of the earth's radius."

**Solution by the Editor.**

The solidity of an oblate spheroid = \( \frac{4}{3} \pi a^2 b \), and its surface

\[ 2\pi a^2 \left[ 1 + \frac{\epsilon^3}{2 \epsilon} \log \left( \frac{1+\epsilon}{1-\epsilon} \right) \right] \]

Let \( \rho_1, \rho_2, \rho_3, \ldots, \rho_n \) represent the radius at \( n \) different points of the earth's surface, and let \( s \) represent one \( n^{th} \) part of the surface; then, if \( n \) is increased without limit, we shall have

\[ s \cdot \frac{4}{3}(\rho_1 + \rho_2 + \rho_3 + \ldots + \rho_n) = \frac{4}{3} \pi a^2 b \ldots \ldots \ldots \ldots (1) \]

But \( (\rho_1 + \rho_2 + \rho_3 + \ldots + \rho_n) + n \) is the mean radius, and

\[ ns = 2\pi a^2 \left[ 1 + \frac{\epsilon^3}{2 \epsilon} \log \left( \frac{1+\epsilon}{1-\epsilon} \right) \right] \ldots \ldots \ldots \ldots (2) \]

By dividing the two members of (1) respectively by the corresponding members of (2) we get

\[ \frac{1}{3}(\rho_1 + \rho_2 + \rho_3 + \ldots + \rho_n) = \frac{2b}{3 \left[ 1 + \frac{\epsilon^3}{2 \epsilon} \log \left( \frac{1+\epsilon}{1-\epsilon} \right) \right]} \]

or

\[ (\rho_1 + \rho_2 + \rho_3 + \ldots + \rho_n) = \frac{2b}{\left[ -1 + \frac{\epsilon^3}{2 \epsilon} \log \left( \frac{1+\epsilon}{1-\epsilon} \right) \right]} \]

= the mean radius, nearly.

This solution is rigorously exact only on the supposition that the radii \( \rho_1, \rho_2, \rho_3, \ldots \) are everywhere perpendicular to \( s \), which is only the case,
however, at the equator and poles. The maximum angle of the vertical, which occurs at latitude 45°, is, however, only about 11', so that the solution gives a close approximation.

A solution of the question, not restricted to the earth, but that shall apply to any ellipsoid of revolution, can not be effected without resort to series, or elliptic functions, as the following solution will show.

SOLUTION BY ARTEMAS MARTIN, ERIE, PA.

Let \( a \) be the equatorial and \( b \) the polar radius of the earth, then \( a^2 - b^2 = a^2 e^2 \), and

\[
\begin{align*}
b &= a\sqrt{1 - e^2}. \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \quad (1)
\end{align*}
\]

Let \( R \) be any radius-vector drawn from the center of the earth to its surface; \( x \) and \( y \) the corresponding co-ordinates, the origin being at the center of the earth; \( S \) the surface of the earth, and \( \Delta \) the mean radius; then

\[
\begin{align*}
R &= \sqrt{x^2 + y^2} \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \quad (2), \quad \text{and} \quad \Delta = \frac{\int R\,dS}{\int dS} \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \quad (3)
\end{align*}
\]

Assuming the earth to be an oblate spheroid of revolution, the equation to the generating ellipse is

\[
\begin{align*}
a^2 y^2 + b^2 x^2 &= a^2 b^2 \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \quad (4), \quad \text{or, by (1),} \quad y^2 + (1 - e^2)x^2 &= a^2 (1 - e^2) \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \quad (5)
\end{align*}
\]

and

\[
\begin{align*}
dS &= 4\pi x \sqrt{dx^2 + dy^2} = \frac{4\pi}{1 - e^2} \sqrt{a^2 (1 - e^2)^2 + e^2 y^2} dy \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \quad (6)
\end{align*}
\]

Integrating (6) between the limits \( y = a\sqrt{1 - e^2} \) (= \( a'u \)), \( y = 0 \),

\[
\begin{align*}
S &= 2\pi a^2 + \frac{\pi a^2 (1 + e^2)}{e} \log \frac{1 + e}{1 - e} \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \quad (7)
\end{align*}
\]

By substitution (3) becomes

\[
\begin{align*}
\Delta &= \frac{4\pi}{S(1 - e^2)^{1/2}} \int_0^1 [a^2 (1 - e^2) - e^2 y^2]^{1/2} [a^2 (1 - e^2) + e^2 y^2]^{1/2} dy \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \quad (8)
\end{align*}
\]

Put \( a^2 (1 - e^2) - e^2 y^2 = w, \) then \( y = (1 + e) \sqrt{a^2 (1 - e^2) - w^2} \) and

\[
\begin{align*}
\Delta &= \frac{4\pi}{S e(1 - e^2)^{1/2}} \int_{a^2 w(1 - e^2)}^{a^2 w(1 - e^2)} \frac{\sqrt{a^2 (1 - e^2) (1 - e^2) - w^2}}{\sqrt{a^2 (1 - e^2) - w^2}} dw.
\end{align*}
\]

Let \( a\sqrt{1 - e^2} = w, \) and \( c^2 = 1 - (2 - e^2), \) then \( dw = a(1 - e^2)^{1/2} dv \) and

\[
\begin{align*}
\Delta &= \frac{4\pi a^3 (2 - e^2)^{1/2}}{e S} \int_0^1 \frac{\sqrt{1 - c^2 v^2}}{\sqrt{1 - v^2}} dv \\
&= \frac{4\pi a^3}{o e S} \left[ \int_0^1 (1 - c^2 v^2)^{3/2} dv - \int_0^1 (1 - (1 + c^2 v^2 + c^2 v^4) dv - \int_0^1 (1 - c^2 v^2)(1 - c^2 v^4)^{3/2} dv \right].
\end{align*}
\]

\[
\begin{align*}
&= \frac{4\pi a^3}{o e S} \left[ \int_0^1 (1 - c^2 v^2) dv - \int_0^1 (1 - 2(1 + c^2 v^2 + 3 c^2 v^4) dv - \int_0^1 (1 - c^2 v^2)(1 - c^2 v^4)^{3/2} dv \right].
\end{align*}
\]
But \[
\int \frac{2-(1+\psi^2)\psi}{(1-\psi^2)^3(1-\psi^2)^2} d\psi = (1+\psi^2) \int \frac{(1-\psi^2)^4 d\psi}{(1-\psi^2)^3(1-\psi^2)^2} - (1-\psi^2) \int \frac{d\psi}{(1-\psi^2)^3(1-\psi^2)^2},
\]
and
\[
\int \frac{1-2(1+\psi^2)\psi + 3\psi^4}{(1-\psi^2)^3(1-\psi^2)^2} d\psi = \psi \sqrt{(1-\psi^2)/(1-\psi^2)};
\]

\[
\therefore \quad \Delta = \frac{4\pi \alpha_3}{c o S} \left[ \frac{1}{2} (2-\psi^2) \int (1-\psi^2)^3 d\psi + \frac{1}{2} (1-\psi^2) \int (1-\psi^2)^3 (1-\psi^2)^2 d\psi \right.
\]
\[
- \frac{1}{2} \psi \sqrt{(1-\psi^2)/(1-\psi^2)} \right],
\]

\[
= \frac{4\pi \alpha_3}{3 c o S} \left[ c \psi (1-\psi^2) + (2-\psi^2) \left[ E'(c) \right] + (1-\psi^2) \left[ F'(c) \right] \right],
\]

where \( F' \) and \( E' \) denote elliptic functions of the first and second orders.

140. “A point is taken at random within a given circle, and a random chord drawn through it. If another chord be drawn at random, what is the probability that it will intersect the first?”

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

Let \( P \) be the random point, \( O \) the center of the circle, \( AB \) the random chord through \( P \), and \( CD \) the second random chord. Draw \( OH \) perpendicular to \( AB \). Put \( OP = x \), \( OA = 1 \), \( \angle OPA = \theta \), \( \angle AOH = \phi \).

Then we have \( OH = x \sin \theta = \cos \phi \), \( x = \csc \theta \cos \phi \), \( dx = - \csc \theta \sin \phi \, d\phi \). The second random chord may be drawn from any point of the circumference to any other point; if it be drawn from any point of either of the arcs \( ACB, ADB \) to any point of the other, it will intersect the first. Hence while \( P \) and \( AB \) are fixed, the number of ways \( CD \) can be drawn to intersect \( AB \) is \( 2 \) \( ACB \times ADB = 8 \phi (\pi - \phi) \); and the whole number of ways it can be drawn is \( 4\pi^3 \). The limits of \( x \) are 0 and 1, of \( \theta \), 0 and \( \frac{1}{2} \pi \).

Hence the required probability is

\[
p = \frac{\int_{\frac{1}{2}}^{\pi} \int_{\theta}^{\pi} 8 \phi (\pi-\phi) d\theta \times 2 \pi dx}{\int_{\frac{1}{2}}^{\pi} \int_{\theta}^{\pi} 4 \pi^3 d\theta \times 2 \pi dx} \]
\[
= \frac{8 \pi}{\pi} \int_{\frac{1}{2}}^{\pi} \int_{\theta}^{\pi} \phi (\pi-\phi) d\theta \, dx.
\]
\[
\frac{8}{\pi^3} \int_0^{\pi} \int_{\pi-\theta}^{\pi} \varphi(\pi-\varphi) \csc^2 \varphi d\varphi \sin \varphi \cos \varphi d\varphi
\]
\[
= \frac{1}{\pi^3} \int_0^{\pi} (\pi^2 - 4\theta^2 + 2\theta^2 \csc^2 \theta - 4\theta \cot \theta + 2) d\theta
\]
\[
= \frac{1}{3} + \frac{1}{\pi^3}.
\]

**QUERY.** "In Thompson and Tait's 'Elements of Natural Philosophy' it is stated that, 'at the southern base of a hemispherical hill radius \(a\) and density \(\rho\), the true latitude (as measured by the aid of the plumb-line) is diminished by the attraction of the mountain by the angle \(\frac{3}{4} \rho \pi a - (G - \frac{4}{3} \rho a)\), where \(G\) = the attraction of the earth in same units.' How is this proved?"

**ANSWER BY HENRY HEATON.**

As the angle is so small that it is practically equal to its tangent, if \(G =\) attraction of the earth, \(\frac{3}{4} \rho \pi a =\) horizontal, and \(\frac{4}{3} \rho a =\) vertical attraction of the hill, then it is evident that the angle of deflection of the plumb-line = \(\frac{3}{4} \rho \pi a + (G - \frac{4}{3} \rho a)\).

Where the unit of attraction is the attraction of a mass whose volume is 1, density 1, and distance 1, it has been shown, page 194, Vol. II, of the ANALYST, that the horizontal attraction of the hill = \(\frac{3}{4} \rho \pi a\); and that the vertical attraction = \(\frac{4}{3} \rho a\), I shall now proceed to show.

Taking the origin of co-ordinates at the center of the spherical surface, the attraction exerted by an element of the mass resolved in a vertical direction = \(\rho d\xi dy dz\). Hence the vertical attraction of the hill is

\[
\rho \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\xi dy dz = \rho \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \log \left( \frac{\sqrt{2a + \sqrt{(a-x)^2}}}{\sqrt{2a + \sqrt{(a-x)^2}}} \right) - \frac{2\sqrt{(a-x)}dx}{\sqrt{2a}} \right]
\]

\[
= \frac{4}{3} \rho a; \text{ where the limits of integration, } u \text{ and } v, \text{ are, respectively, } u = \sqrt{(a^2 - x^2)} \text{ and } v = \sqrt{(a^2 - x^2 - y^2)}.
\]

If we put \(r\) for radius of the earth, and \(\delta\), its mean density, \(G = \frac{4}{3} \delta r\). Substituting this for \(G\) in the expression for the angle of deflection, we get, \(\rho \pi a - 2(\delta \pi a - \rho a)\). As \(a\) is very small in comparison with \(r\) the expression for the deflection is, practically, \(\rho a - 2\delta r\).

[Mr. Adcock obtains precisely the same result as the last above written, viz.; \(\rho a + 2\delta r\).]
141. By William Hoover, Bellefontaine, Ohio. — Let $ABC$ be any plane triangle, $R$ the radius of its circumscribing circle, $r'$ the radius of the escribed circle opposite $A$, $p_1, p_2, p_3$ the perpendiculars from the center of the circumscribing circle upon $a, b$ and $c$. show that $r' - R = p_2 + p_3 - p_1$.

142. By Marcus Baker, Washington, D. C. — In a plane triangle $ABC$, $AO'$ is drawn bisecting the angle $A$; from $O'$, the center of the escribed circle, a perpendicular to $AO'$ is drawn, meeting $AB$ produced in $T''$, and from $T''$ a perpendicular to $AB$ is drawn, meeting $AO'$ produced in $O''$; with $O''$ as center and $O''T''$ as radius a circle is described; prove that this circle is tangent to the circle circumscribed about the triangle $ABC$.

143. From Exam'n Problems, Hamilton College. — Required the area of the triangle formed by the tangent to the curve whose equation is $x^4 + y^4 = r^4$ and the axes of $xy$.

144. By O. H. Merrill, Rodman, N. Y. — A cylindrical post, diameter $d$, stands perpendicularly on a level plane with a rope, diameter $d'$ and length $l$, attached to the bottom of the post and lying stretched on the plane. How far will a person walk who takes hold of the end of the rope and winds it around the post by walking around it always keeping the rope parallel to the plane and also keeping the coils on the post at a distance of $3d'$ from each other?

145. By Christine Ladd, Union Springs, N. Y. — The product of the lengths of tangents from the radical center of three circles on any pair of circles through the intersections of the given circles is equal to the product of the lengths of tangents from the same point on any pair of circles tangent to the given circles.

146. By D. J. McAdam, Washington, Pa. — Given a semicircle and a circle, place the latter so that it will cut the former; what is the probability that its center will fall within the former?

147. By E. B. Seitz, Greenville, Ohio. — Two rods of equal length have their middle points connected by a string of half the length of one of the rods. If they be thrown on a level floor, what is the chance of their crossing?
148. By Artemas Martin, Erie, Pa. — A tortoise, whose shell is circular, radius $a$, is moving in a straight line at the uniform speed of $n$ feet per minute, and a fly is running around on the edge of its shell at the uniform rate of $n$ feet per minute.

Required the equation to the curve the fly describes in space.

149. From Todhunter's Integral Cal., p. 164, by request of H. Heaton.—Prove that

$$
\int_{\frac{a}{2}}^{x} \frac{dx}{\sqrt{(2ax-x^2)(a^2-x^2)}} = \frac{2}{3a} F\left(\frac{c\pi}{2}\right), \text{where} \ c = \frac{1}{6}.
$$

150. From the Journal of Progress. — An underwriter insures three vessels, the first an iron steamer, the second a steamer not of iron and the third a sailing vessel, at $20,000$, $15,000$, and $10,000$, respectively. One of them is known to have been burned at sea; and three persons, $A$, $B$, $C$, whose respective veracities are $\frac{1}{4}$, $\frac{1}{4}$, and $\frac{1}{6}$, report as follows: $A$, that the lost vessel was an iron steamer; $B$, that it was not a sailing vessel; and $C$, that it was a sailing vessel. Required the expectation of loss to the underwriter, the a priori probability of destruction by fire being twice as great in case of a steamer as of a sailing vessel.

BOOK NOTICES.


In this book the author presents the Theory of equations, Solution of equations by general formulas, Sturm's Theorem, Horner's method of approximation, an analysis of equations by Fourier's theorem, &c., in a concise and lucid manner.

Students of Algebra will not fail to be interested and instructed from a perusal of this book.

Interpolation and Adjustment of Series, By E. L. De Forest. New Haven. 1876.

This is a pamphlet of 50 pages and is a continuation of two other papers on the same subject which were published in the Annual Reports of the Smithsonian Institution, for the years 1871 and 1872.

ERRATUM.

On page 172, (Vol. III) line 6, for Put $r + x = a\sqrt{a}$, read, Put $r = a\sqrt{a}$.
ON THE HISTORY OF THE METHOD OF LEAST SQUARES.

BY MANSFIELD MERRIMAN, PH. D., NEW HAVEN, CT.

It is the object of this article to present some brief notes concerning the various demonstrations of the method of Least Squares, with references to the original works or memoirs in which they were given.

The honor of the first publication of the method belongs to Legendre. It is given in his work *Nouvelles méthodes pour la determination des orbites des comètes*, published at Paris in 1805. It is a quarto of viii + 80 pages: some copies have a supplement and a title page dated 1806. In this work the rule that the sum of the squares of the errors should be made a minimum to obtain the adjusted values of observed quantities is proposed as a convenient method only. The rule for the formation of normal equations is deduced and applied in practical examples, the arithmetical mean is shown to be a particular case of the method and the analogy of the formula with those derived in mechanics for finding the centre of gravity of bodies is noticed. Referring to this analogy Legendre says: "la méthode des moindres carrés fait connoître, en quelque sorte, le centre autour duquel viennent se ranger tous les résultats fournis par l'expérience, de manière à s'en écarter le moins qu'il possible." Although this can scarcely be called a proof, it shows that Legendre fully appreciated the advantage of the method.

The first proof of the method of Least Squares was given by Dr. Robert Adrain in *The Analyst*, an American Journal edited by him during the early part of this century. The title of his paper is *Research concerning the probabilities of the errors which happen in making observations*, and it is given in *The Analyst* for 1808, No. IV, pp. 93 — 109. The term "Least Squares" is not used and Adrain seems to have been entirely unacquainted with Legendre's researches. The proof consists in showing that the probability $y$ of the error $x$ is given by an equation of the form
\[ y = e^{-\frac{1}{2}z^2} \]

in which \( c \) and \( h \) are constants and \( e \) the base of the natural system of logarithms. Hence by well known rules the probability of the system of errors \( x_1, x_2, \ldots, x_n \) is

\[ Y = c^e^{-h^2(x_1^2 + x_2^2 + \ldots + x_n^2)} \]

and the most advantageous or most probable system will be that for which \( Y \) is a maximum, and this requires that

\[ x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2 = \text{a minimum} \]

whence the method of Least Squares. The proof by which the law of facility or probability of error is established cannot be discussed here: it is analysed and shown to be unsatisfactory by Glaisher in the Memoirs . . . . . . Astron. Soc. London, Vol. XXXIX, pp. 75—81. Adrain's paper was unknown to mathematicians until 1871, when it was republished in Amer. Jour. Sci., Vol. I, pp. 412—414.

The second proof was given by Gauss in the Theoria motus corporum coelestium . . . . published at Hamburg in 1809. It occupies pages 205—224 of the volume. Assuming that the arithmetical mean is the most probable value of a quantity directly observed, Gauss shows that the law of facility of error must be of the exponential form

\[ y = c e^{-\frac{1}{2}z^2} \]

from which the rule of Least Squares follows. This is the proof given in the great majority of books. The assumption of the rule of the arithmetical mean impairs greatly its strictness, and as presented in Chauvenet's treatise (Appendix to Astronomy) it is particularly illogical, the constant \( c \) being given the value \( h \pi^{-\frac{1}{2}} \), so that the probability of an error \( x_1 \) is a finite quantity. The constant \( c \) should be \( h \pi^{-\frac{1}{2}} dx \).

The third proof was given by Laplace in 1810 in the Memoires . . . Institut France, for 1809, pp. 383 — 389 and 559 — 565, and is reproduced in his Theorie analytique des Probabilités, Chap. IV. It uses only the axiom that positive and negative errors are equally probable and is independent of the particular form of the law of facility of error, provided the number of observations is infinite. It is the most satisfactory of all the demonstrations of the method. For an excellent account and simplification of Laplace's reasoning see Glaisher, Memoirs . . . . . . Astron. Soc. London, Vol. XXXIX, p. 92, et. sq. The proof is given by De Morgan, Airy and other English writers.

On pages 318—319 of the Theorie . . . des Prob. is given what is sometimes called Laplace's second demonstration. It depends on the axiom that
the mean of all the errors taken positively must be a minimum to give the most advantageous values.

The fifth proof was given by Gauss in a memoir entitled: *Theoria combinationis observationum erroribus minimis obnoxiae* which was published in 1823 in the *Comment. Soc. Gottingen*, Vol. V, pp. 33—90 and also issued separately. The proof takes for granted that the mean value of the sum of the squares of the errors may be used as a measure of the precision of the observations. It introduces no law of facility of error. The proof is entirely untenable and has we believe only been followed by Helmert in 1872.

The sixth proof was given by Ivory in 1825 in an article *On the Method of the Least Squares* in Tilloch's *Phil. Mag.*, Vol. LXV, pp. 3—10. It rests on a vague analogy with the properties of the lever, and was well refuted by Ellis in *Trans. Camb. Phil. Soc.*, Vol. VIII, pp. 217—219.

The seventh attempted proof was given by Ivory in 1826 in Tilloch's *Phil. Mag.*, Vol. LXVIII, pp. 161—165. Although quoted by Françoiseur in his *Astronomie pratique* (Paris 1830), pp. 426—428, as perfectly valid, it is in our opinion the most unsatisfactory of all. See also Ellis' paper quoted above.

The eighth proof was given in 1837 by Hagen in his *Grundzüge der Wahrscheinlichkeitsrechnung* published at Berlin in octavo. The proof rests on the hypothesis that an error is the algebraic sum of an indefinitely large number of small elementary errors, which are all equal and which are equally likely to occur as positive or negative. This postulated, Hagen proves by the law of combinations that the law of facility of error is

\[ y = ce^{-a^2x^2} \]

whence the rule of Least Squares as shown above. Next to Laplace's demonstration this is to us the most satisfactory proof. It is given by Wittstein in 1849, by Dienger in 1852, by Encke in 1850 (see *Berliner Jahrbuch* for 1853), and in a slightly modified form by Quetelet, Tait and others. The "New Investigation..." in this *Journal* for Sept. 1876, also gives Hagen's proof on p. 133 — 135. A second edition of Hagen's book, we may mention, appeared in 1867.

The ninth proof is due to Bessel and was published in an article entitled *Untersuchungen uber die Wahrscheinlichkeit der Beobachtungsfehler* which appeared in the *Astronomische Nachrichten*, 1838, Vol. XV, col. 369—404. It considers errors as arising from sources of error, which need not as in Hagen's proof be very small. The result is that the law of error approximates closely to the exponential form when many sources of error act together.
The tenth proof was given in 1844 by Donkin in An essay on the Theory of the Combination of Observations printed at Oxford, and in 1855 translated in Liouville's Jour. Math., Vol. XV, pp. 297—322. The usual rules for the adjustment of observations are here deduced by a kind of metaphysical statics from the laws of Mechanics. No law of facility is employed.

The eleventh proof was given in 1850 by John Herschel in a review, Quetelet on Probabilities, in the Edinburgh Review, Vol. XCII, pp. 1—57. It deduces the exponential law of facility by reasoning which is not received by the best mathematicians. See Glaisher's paper quoted above, and Ellis in Lond. Phil. Mag., 1850, Vol. XXXVII, pp. 321—328.


A thirteenth investigation which may be mentioned is by Croston, On the Proof of the Law of Errors of Observations in the Phil. Trans..., London for 1870, pp. 175—188, in which the law of facility of error is investigated, on the hypothesis that an error arises from the joint operation of a large number of small errors, positive and negative errors not being equally probable a priori. The discussion is not very clear.

Although no one of these thirteen proofs is perfectly satisfactory, yet each is valuable as setting the subject in a new light and illustrating the laws of thought which have led to the universal employment of the arithmetical mean, and other rules for the adjustment of observations. No discussion of their relative values can be attempted here. The critical papers of Ellis and of Glaisher, quoted above, may in this connection be mentioned as of great value to students of the theory of the subject.

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ON THE ROTATION OF SATURN.

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BY PROF. A. HALL.

On the night of December 7th while observing Japetus I noticed a bright spot on the ball of Saturn. The spot was 2' or 3' in diameter, round and well defined, and of a brilliant white color. It was north of the Ring, nearly midway of the disk in the direction of a circle of declination, and when first seen was about one third the distance from the center of the Ball toward the following edge. The transit of the spot over the center of the disk or rather over the line bisecting the disk and perpendicular to the major
axis of the Ring, occurred at 6° 18′, Washington m. t. On the next day letters were sent by Admiral Davis to astronomers in different parts of the country asking them to assist in observing the appearance of the spot. With these letters was sent an ephemeris of the spot, computed by assuming the time of the rotation of Saturn to be

10° 29′ 16′:8 m. t.

which is given in nearly all the modern text books as Sir W. Herschel’s last and most accurate determination. Although this time of rotation proved to be erroneous, and thus the ephemeris was, perhaps, worse than useless, fortunately the spot was observed on Dec. 10th by Professor Maria Mitchell, at the Vassar College Observatory, Poughkeepsie, New York; by Mr. Lewis Boss at the Dudley Observatory, Albany; by Mr. D. W. Edgecomb at Hartford, Connecticut, and by Messrs Alvin Clark and Sons at Cambridgeport, Massachusetts. The spot was observed at Washington until Jan. 2, when having become faint and indistinct, and the weather being bad and the position of the planet unfavorable, the observations were given up. The following table gives the dates and the Washington mean time when the spot was observed to be central on the disk of the planet, the place of observation, the observer and his remarks, and the weight of the observation.

<table>
<thead>
<tr>
<th>Date</th>
<th>Place</th>
<th>Observer</th>
<th>Wt.</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dec. 7 6°18′</td>
<td>Washington</td>
<td>A. Hall</td>
<td>1.00</td>
<td></td>
</tr>
<tr>
<td>10 6 6</td>
<td>Cambridgeport</td>
<td>A. G. Clark</td>
<td>0.50</td>
<td></td>
</tr>
<tr>
<td>10 6 3</td>
<td>Hartford</td>
<td>D.W. Edgecomb</td>
<td>0.50</td>
<td>The spot is oval; its brilliancy is twice that of the equatorial belt.</td>
</tr>
<tr>
<td>10 6 4</td>
<td>Poughkeepsie</td>
<td>M. Mitchell</td>
<td>0.50</td>
<td>The spot very bright at its center; elliptical in form.</td>
</tr>
<tr>
<td>10 2</td>
<td>Albany</td>
<td>L. Boss</td>
<td></td>
<td>Spot not seen certainly till past cent.</td>
</tr>
<tr>
<td>13 5 47</td>
<td>Washington</td>
<td>A. Hall</td>
<td>1.00</td>
<td></td>
</tr>
<tr>
<td>13 5 50</td>
<td>&quot;</td>
<td>J. R. Eastman</td>
<td>0.50</td>
<td></td>
</tr>
<tr>
<td>13 5 38</td>
<td>Cambridgeport</td>
<td>A. G. Clark</td>
<td>0.50</td>
<td></td>
</tr>
<tr>
<td>16 5 31</td>
<td>Washington</td>
<td>A. Hall</td>
<td>0.50</td>
<td>Extremely poor.</td>
</tr>
<tr>
<td>16 5 6</td>
<td>Cambridgeport</td>
<td>A. G. Clark</td>
<td>0.25</td>
<td>Very bad seeing.</td>
</tr>
<tr>
<td>19 5 2</td>
<td>Washington</td>
<td>J. R. Eastman</td>
<td>0.50</td>
<td></td>
</tr>
<tr>
<td>19 5 6</td>
<td>&quot;</td>
<td>A. Hall</td>
<td>1.00</td>
<td></td>
</tr>
<tr>
<td>19 5 7</td>
<td>&quot;</td>
<td>S. Newcomb</td>
<td>1.00</td>
<td></td>
</tr>
<tr>
<td>19 4 48</td>
<td>Cambridgeport</td>
<td>A. G. Clark</td>
<td>0.25</td>
<td>Very bad seeing.</td>
</tr>
<tr>
<td>21 8 19</td>
<td>Washington</td>
<td>A. Hall</td>
<td>1.00</td>
<td></td>
</tr>
<tr>
<td>27 7 35</td>
<td>&quot;</td>
<td></td>
<td>1.00</td>
<td></td>
</tr>
<tr>
<td>27 7 42</td>
<td>&quot;</td>
<td>J. R. Eastman</td>
<td>0.50</td>
<td></td>
</tr>
<tr>
<td>30 7 25</td>
<td>&quot;</td>
<td>A. Hall</td>
<td>1.00</td>
<td></td>
</tr>
<tr>
<td>Jan. 2 7 9</td>
<td></td>
<td></td>
<td>1.00</td>
<td></td>
</tr>
</tbody>
</table>
The observations at Washington with the large refractor were made by placing the wire of the micrometer perpendicular to the major axis of the Ring, and, with the aid of the driving clock, bisecting the disk of the planet with the wire; and in this way estimating the time when the spot came to the center of the disk. Professor Eastman observed with the 9½ inch refractor. The observation by Professor Newcomb, Dec. 19, was made with the large refractor, and we endeavored to be independent in our determination of the time. In assigning the weights I have given to an observation with the large refractor the weight unity, and to all the others a weight of one half, except in a few cases where the notes require the weights to be diminished. Toward the latter part of the observations the spot had become elongated on the preceding side into a bright belt, and for this reason the observations made during this time are not so good as those made earlier, but I have not changed the weights on this account. In the following table the times have been corrected for the aberration of light, and for convenience the different dates are designated by the letters a, b, c, d &c. The weights are given in the column p.

<table>
<thead>
<tr>
<th></th>
<th>Wash. m. t.</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>1876, Dec. 7</td>
<td>4º 55.7&quot;</td>
<td>1.00</td>
</tr>
<tr>
<td>&quot; 10</td>
<td>4</td>
<td>41.6</td>
</tr>
<tr>
<td>&quot; 13</td>
<td>4</td>
<td>22.5</td>
</tr>
<tr>
<td>&quot; 16</td>
<td>3</td>
<td>59.3</td>
</tr>
<tr>
<td>&quot; 19</td>
<td>3</td>
<td>40.0</td>
</tr>
<tr>
<td>&quot; 21</td>
<td>6</td>
<td>55.0</td>
</tr>
<tr>
<td>&quot; 27</td>
<td>6</td>
<td>12.5</td>
</tr>
<tr>
<td>&quot; 30</td>
<td>5</td>
<td>59.8</td>
</tr>
<tr>
<td>1877, Jan. 2</td>
<td>5</td>
<td>43.6</td>
</tr>
</tbody>
</table>

In order to deduce from these observations the time of the rotation of Saturn the following method has been used. We have given the position in space of the two right lines, the lines of sight, from the observer to the spot on any two dates and hence the angle between these lines can be computed. It is assumed that the angle lies in the plane passing through the positions of the Earth for the two dates and the position of Saturn for the mean of these dates; also that the vertex of this angle is at the center of Saturn. If we denote by i the inclination of the equator of Saturn to the ecliptic, and by N the longitude of its ascending node: also by i' and N' similar quantities for the plane of the lines of sight, the inclination J of the plane of the equator of Saturn to the plane of the lines will be given by the equation
\[ \cos J = \cos i \cos i' - \sin i \sin i' \cos (N - N') \]

where \( i = 28^\circ \, 10', \) and \( N = 167^\circ \, 53'. \)

If \( \theta \) be the angle between the two lines for any two dates, and \( \varphi \) this angle reduced to the equator of Saturn we shall have

\[ \tan \varphi = \tan \theta \cos J. \]

If we denote by \( a_1, \beta_1, r_1; \) \( a_2, \beta_2, r_2, \) the coordinates of the Earth for the first and second dates; and by \( a_3, \beta_3, r_3, \) the coordinates of Saturn for the mean of the dates: and if we represent the equation of the plane passing through these points by

\[ Ax + By + Cz + D = 0 \]

we shall have

\[ A = (\beta_2 - \beta_1)r_3: \quad B = (a_1 - a_2)r_3 \]
\[ C = a_1\beta_2 - a_2\beta_1 + a_2\beta_3 - a_3\beta_2 + a_3\beta_1 - a_1\beta_3: \]

and

\[ \sin i' = \frac{\sqrt{(A^2 + B^2)}}{\sqrt{(A^2 + B^2 + C^2)}}: \quad \tan N' = -\frac{A}{B}. \]

As the observations show that the time of rotation is nearly \( 10^h \, 15', \) if we express \( \varphi \) in minutes of arc the correction to the observed interval, in minutes of time, will be

\[ \Delta t = \frac{10.25}{360}, \quad \varphi' = (8.4544)\varphi'. \]

By this method the observed times have been corrected for the motions of the Earth and Saturn, and made comparable with the observation of December 7th.

I now assume Dec. 7.0 as the epoch, and that \( m \) is the time from this epoch to the first observation: then if \( t_n \) be any observed time, \( n \) the number of rotations of Saturn after the epoch and \( r \) the time of one rotation we have

\[ nr + n = t_n. \]

Putting \( r = r_0 + x; \quad m = m_0 + y; \)

\[ nr_0 + m_0 - t_n = \Delta, \]

where \( r_0 \) and \( m_0 \) are approximate values, the equation of condition is

\[ \sqrt{\rho}.(nx + y + \Delta) = 0. \]

The assumed approximate values are

\[ r_0 = 614^m.5; \quad m_0 = 296^m.0. \]
The following table gives the angle $\theta$, in minutes of arc, for each of the positions with respect to the first; the values of $i'$ and $N'$, the correction $\Delta t$, to the observed time, $n$ the number of rotations of Saturn from the position a, the residual $\Delta$ and its weight, and in the last column the values of the residuals found by substituting the resulting values of $x$ and $y$ in the equations of condition.

<table>
<thead>
<tr>
<th>Position</th>
<th>$\theta$</th>
<th>$i'$</th>
<th>$N'$</th>
<th>$\Delta t$</th>
<th>$n$</th>
<th>$\Delta$</th>
<th>$p$</th>
<th>$\sigma/\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>0.00'</td>
<td>—</td>
<td>—</td>
<td>-0.00</td>
<td>0</td>
<td>+0.30</td>
<td>1.00</td>
<td>+2.18</td>
</tr>
<tr>
<td>b</td>
<td>9.55</td>
<td>6°02'</td>
<td>349°07'</td>
<td>0.25</td>
<td>7</td>
<td>-3.85</td>
<td>1.50</td>
<td>-3.31</td>
</tr>
<tr>
<td>c</td>
<td>22.32</td>
<td>5</td>
<td>43</td>
<td>350 05</td>
<td>0.59</td>
<td>14</td>
<td>-2.91</td>
<td>2.00</td>
</tr>
<tr>
<td>d</td>
<td>33.60</td>
<td>5</td>
<td>24</td>
<td>351 15</td>
<td>0.88</td>
<td>21</td>
<td>+2.08</td>
<td>0.75</td>
</tr>
<tr>
<td>e</td>
<td>46.77</td>
<td>5</td>
<td>0</td>
<td>352 53</td>
<td>1.23</td>
<td>28</td>
<td>+3.23</td>
<td>2.75</td>
</tr>
<tr>
<td>f</td>
<td>55.40</td>
<td>4</td>
<td>50</td>
<td>353 18</td>
<td>1.45</td>
<td>33</td>
<td>+0.95</td>
<td>1.00</td>
</tr>
<tr>
<td>g</td>
<td>85.35</td>
<td>4</td>
<td>16</td>
<td>356 43</td>
<td>2.22</td>
<td>47</td>
<td>+7.22</td>
<td>1.50</td>
</tr>
<tr>
<td>h</td>
<td>101.03</td>
<td>4</td>
<td>2</td>
<td>358 17</td>
<td>2.62</td>
<td>54</td>
<td>+1.82</td>
<td>1.00</td>
</tr>
<tr>
<td>i</td>
<td>115.90</td>
<td>3</td>
<td>51</td>
<td>359 45</td>
<td>-3.00</td>
<td>61</td>
<td>-0.10</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Hence we have the following equations of condition:

$$
0.00 \times + 1.00 \, y + 0.30 = 0 \\
8.57 \times + 1.22 \, y - 4.72 = 0 \\
19.80 \times + 1.41 \, y - 4.12 = 0 \\
18.17 \times + 0.87 \, y + 1.80 = 0 \\
46.44 \times + 1.66 \, y + 5.36 = 0 \\
33.00 \times + 1.00 \, y + 0.95 = 0 \\
57.56 \times + 1.22 \, y + 8.84 = 0 \\
54.00 \times + 1.00 \, y + 1.82 = 0 \\
61.00 \times + 1.00 \, y - 0.10 = 0 
$$

The solution of these equations by the method of least squares gives the following values of $x$ and $y$.

The normal equations being

$$
+ 13991.43 \times + 349.50 \, y + 791.95 = 0 \\
+ 349.50 \times + 12.50 \, y + 12.68 = 0 
$$

we have

$$
x = -0^m.1037 \pm 0^m.0383; \quad y = +1^m.88 \pm 1^m.282.
$$
The probable error of an observation of weight unity is \( \pm 2^m.49 \). Hence the mean time in which Saturn rotates is

\[ 10^b \ 14^m \ 23^s.8 \pm 2^s.30 \]

This value has been found by assuming that the spot had no proper motion on the surface of the planet; whether this is really the case or not we cannot determine from the observations.

The only determination of this rotation, so far as I know, is the one by Sir William Herschel, Philosophical Transactions, London, 1794, page 48. By observing the different appearances of a quintuple belt from Nov. 11th 1793, until Jan. 16th, 1794. Herschel found the time of rotation to be, in mean time,

\[ 10^b \ 16^m \ 0^s.4 \]

and concludes that this time cannot be in error by so much as two minutes. At the time of the discovery of the white spot this determination of Herschel was not known to me; and as stated before, the time of rotation was assumed to be

\[ 10^b \ 29^m \ 16^s.8 \]

and through this mistake several observers failed to see the spot. This last time is given by nearly all the astronomical writers, and by several, Proctor, Klein, Wolf &c., it is said to be Herschel's final determination; "schlieslich zu 10 Stunden 29 Minuten 17 Secunden be stimmte." But I have not found that Sir William Herschel ever gave this value. As to its origin I cannot speak with certainty; but it may have been found in the following manner. In his Système du Monde Laplace says that Saturn rotates in 0.428; and that the Ring rotates in 0.437, these numbers being expressed in decimals of a day. If we reduce them to hours, minutes and seconds we have

\[ 0^h.428 = 10^b \ 16^m \ 17^s.2 \text{ and } 0^h.437 = 10^b \ 29^m \ 16^s.8 \]

The first value given by Laplace is Herschel's, correct to the last decimal of a day and is given in some of the older books. The other value it will be seen is the one commonly given. Hence I think it probable that some one, about fifty years ago, took the wrong number from Laplace and reduced it to hours, minutes and seconds, and that the book makers have faithfully copied this mistake. It should be stated however that one American writer, Professor John Brocklesby, gives Herschel's value very nearly correct, viz., \( 10^b \ 16^m \ 4^s \).

One other mistake on this question may be noticed. In the ninth edition of Sir John Herschel's "Outlines of Astronomy," p. 343, the time of the rotation of Saturn is said to be
as determined by Mr. Airy. But Mr. Airy did not determine the rotation of Saturn but of Jupiter, and the time given is the sidereal time of the rotation of Jupiter, which still needs a small correction on account of aberration. 1877, Jan. 20.

CLASSIFICATION OF PLANE CURVES WITH REFERENCE TO INVERSION.

BY PROF. W. W. JOHNSON, ST. JOHN'S COLLEGE, ANNAPOLIS, MD.

1. The degree of the inverse of a given curve of the nth degree is in general 2n, (thus the inverse of a conic is in general of the 4th degree,) nevertheless it is possible so to classify curves that a given curve and its inverse with reference to any point shall belong to the same family.

The polar coordinates (r', θ') of the point on the inverse curve corresponding to the point (r, θ) of the given curve are \( r' = k^2/r, \quad \theta' = \theta; \) where \( k \) is the modulus of inversion, and the centre of inversion is the pole. Hence denoting the rectangular coordinates by \( (x, y) \) and \( (x', y') \) we derive

\[
\begin{align*}
x &= \frac{k^2 x'}{r'^2} \\
y &= \frac{k^2 y'}{r'^2}.
\end{align*}
\]

Substituting these values for \( x \) and \( y \) in the equation of the given curve we have the equation of the inverse curve, \( r'^2 \) being finally replaced by \( x'^2 + y'^2 \).

2. Now the highest terms of the given equation being denoted by \( u_n \), those of the \((n-1)\)th degree by \( u_{n-1} \) &c.; if \( u_n \) contains a power of \( x^2 + y^2 \) as a factor, let this factor be replaced by the like power of \( r^2 \). Let the terms \( u_{n-1} \) be treated in the same way, and so on with each group of terms of a degree exceeding \( \frac{1}{2}n \). Regarding \( r^2 \) for a moment as a factor of only one dimension, ascertain the degree of the equation in \( r^2 \), \( x \) and \( y \). Call this the circular degree of the equation, and denote it by \( m \). The complete equation of the \( m \)th circular degree may then be written in the annexed tabular form. The terms of like degree are written in the same horizontal line and the degree of the eq'n cannot exceed \( 2m \).

The circular degree of all the terms above the central line is \( m \).
3. The circular degree of an equation is not changed by transformation of coordinates; for, when the direction of the axes is changed without change of origin, \( r^2 \) is unchanged, while for \( x \) and for \( y \) linear functions of \( x \) and \( y \) are substituted; and when the origin is transferred say to the point \((a, b)\), we have to substitute for \( x \), \( x+a \), for \( y \), \( y+b \) and for \( r^2 \) the expression \( r^2 + 2ax + 2by + a^2 + b^2 \), all of which are of the same circular degree with the expressions they replace.

We may therefore classify curves into circular orders according to the circular degrees of their equations. It is obvious that if \( S = 0 \) is the equation of a curve of the \( p \)th circular order, and \( S' = 0 \) that of a curve of the \( q \)th circular order, the circular order of \( SS' = 0 \) will be \( p+q \), and that of \( S + kS' = 0 \) will be \( p \), supposing \( q \) to be not greater than \( p \).

4. A curve and its inverse are of the same circular order. The origin of co-ordinates being taken at the required centre of inversion we obtain the circular equation of the inverse by making in the given equation the substitutions,

\[
x = \frac{k^2x'}{r'^2}, \quad y = \frac{k^2y'}{r'^2}, \quad r^2 = \frac{k^4}{r'^2}.
\]

Let

\[ A x^{\alpha} y^{\beta} (r^2)^{p} \]

represent any term of the given equation of the \( m \)th circular degree. The circular degree of this term, namely \( \alpha + \beta + p \), cannot exceed \( m \); then let

\[ \alpha + \beta + p + q = m. \]

By substitution, the term (1) becomes

\[ A \frac{x'^{\alpha} y'^{\beta} (k^2)^{2\alpha + \beta + 2p}}{(r'^2)^{2\alpha + \beta + 2p}}. \]

There is at least one term in the given equation, in which \( \alpha + \beta + p = m \), hence, to free the equation of the inverse from fractions, it is always necessary to multiply throughout by \((r'^2)^m\); the above term of the inverse then becomes, (omitting the accents),

\[ A x^{\alpha} y^{\beta} (r^2)^{m} (k^2)^{2\alpha + \beta + 2p}. \]

The circular degree of this term is \( \alpha + \beta + q \); and, since \( p = 0 \) in at least one of the terms of the original equation, (otherwise it would have been divisible by \( r^2 \)), the greatest value of \( \alpha + \beta + q \) is \( m \), hence the inverse curve is of the \( m \)th circular order.

5. It will be observed that the exponents of \( x \) and \( y \) are unchanged, and that the sum of the indices of the degree (not circular) of the terms (1) and (2) is \( 2m \), hence we may say that each term is converted into one of complementary degree, with respect to \( 2m \), this conversion being effected by
simply adjusting the exponent of \( r^2 \), and then multiplying by the power of
the modulus indicated by the degree \((\alpha + \beta + 2p)\) of the original term.

Thus the equation of the cissoid is

\[ r^2 x = 2ay^3 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (3) \]

in which \( m = 2 \): taking \( k = 2a \), the inverse is

\[ k^2 x = 2ay^3 k^4 \]
or the parabola

\[ y^2 = 2ax. \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (4) \]

The equation of the equilateral hyperbola is

\[ x^2 - y^2 = a^2. \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (5) \]
taking \( k = a \), the inverse is the lemniscate,

\[ r^4 = (x^2 + y^2)^2 = a^2(x^2 - y^2). \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (6) \]

Again, in the case of the cissoid, if we transfer the origin to the point
\((a, 0)\) [centre of the generating circle] we have the equation

\[ r^2 x + 3ax^3 - ay^4 + 3a^2x + a^4 = 0, \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (7) \]
hence the inverse with respect to this point, and modulus \( a^3 \), is

\[ r^4 + 3r^2ax + 3a^2x^3 - a^2y^6 + a^4x = 0. \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (8) \]

6. In the general equation of the \( m \)th circular degree, as arranged in
Art. 2, the complementary terms are those symmetrically situated with respect to the central row of terms; hence if we make \( k^3 = 1 \), the equation of the inverse curve will be found by simply interchanging \( A \) and \( A', B \) and \( B', \&c. \)

7. The general equation of the \( m \)th circular degree includes every equation of the \( m \)th degree in which the group of terms of highest degree is not divisible by \( x^2 + y^2 \). If these terms are so divisible the equation belongs to a lower circular order. Thus the second circular order includes all conics except the circle, which together with the straight line constitutes the first circular order.

8. Denoting the circular order of the curve by \( m \), and the degree of the equation by \( n \), if \( n = m + 1 \), so that the highest terms of the equation are in the first row above the central row, the curve is said to be a circular \((m+1)\)-ic. If \( n = m + 2 \) the curve is said to be a bi-circular \((m + 2)\)-ic. Thus the second circular order includes besides the non-circular conics, the circular cubics and the bi-circular quartics. In general, the \( m \)th circular order embraces the \( q \)-circular \((m + q)\)-ics, where \( q \) may be called the index of circularity, and the degree of the curve is

\[ n = m + q, \]
in which \( q \) cannot exceed \( m \). Hence, among curves of a given degree, the curves of highest circularity belong to the lowest circular order.
9. The points at infinity in a given curve are found by equating to zero the terms of highest degree; hence to determine these points for the complete equation which represents an $m$-circular $2m$-ic, we have
\[ Ar^{2m} = A(x^2 + y^2)^m = A(x + iy)^m(x - iy)^m = 0, \]
where $i = \sqrt{-1}$. Hence the only points at infinity are the imaginary circular points, and the curve meets the line at infinity $m$ times at each of these points.

The curve has $m$ parallel asymptotes passing through each of these points; for assume the equation
\[ x + iy = \beta, \]
and eliminate $y$ from the equation of the curve by means of this equation. We have
\[ r^2 = 2\beta x - \beta^2, \]
hence the resulting equation will reduce to one of the $m$th degree in $x$, and the coefficient of $x^m$ will obviously contain $\beta^m$. It follows that $m$ values of $\beta$ may be determined, such that the line $x + iy = \beta$ meets the curve in one more point at infinity; in other words, we may determine $m$ parallel asymptotes. The imaginary point at infinity determined by this equation is therefore said to be a multiple point of the order $m$. In like manner the other circular point is a multiple point of the same order.

10. If in the general equation $A = 0$, $q = m - 1$, and the terms of highest degree are
\[ (Bx + Cy)(x^2 + y^2)^{m-1}. \]
The circular points at infinity may now be shown to be multiple points of the order $m - 1$, and there is one real point at infinity determined by the equation
\[ Bx + Cy = 0. \]
In this case, according to Art. 6, the absolute term will vanish from the equation of the inverse curve; hence this curve will pass through the centre of inversion; and, since the terms of the lowest degree in its equation will, by Art. 5, be $(k^2)a^{m-1}(Bx + Cy)$, the above equation will also represent

---

*If the terms of lower degree were not divisible by the powers of $r^2$ as indicated in the general equation (Art. 2), the circular points would not be points of the $m$th order of multiplicity. For instance, if the highest terms of a quartic were $(x^2 + y^2)^3$, and the cubic terms did not contain $(x^2 + y^2)$ as a factor, the result of elimination would be of the third degree, and $\beta$ would not enter the coefficient of $x^3$; hence no asymptotes could be determined. The circular points would not, in that case, be nodes, but points where the curve touches the line at infinity; analogous to the real points corresponding to parabolic branches. The curve instanced would be a case of the circular, not of the bi-circular quartic.
the tangent to the inverse curve at the centre of inversion. It is indeed
evident geometrically that a branch passing to infinity in any direction
inverts into a branch passing in that direction through the centre of inver-
sion. Thus the cissoid, in whose equation, (3) or (7) Art. 5, the term con-
taining $r^4$ is wanting, is a circular cubic having its real point at infinity in
the direction of the axis of $y$; accordingly each of the inverse curves, (4)
and (8), touches the axis of $y$ at the centre of inversion.

11. If in the general equation $A = 0, B = 0$ and $C = 0$, we have
$q = m - 2$; the terms of highest degree then are
$$(Dx^2 + Exy + Fy^2)(x^2 + y^2)^{m-2}.$$ The circular points are now multiple points of the order $m - 2$, and there
are two other points at infinity determined by the equation
$$Dx^2 + Exy + Fy^2 = 0.$$ The terms of lowest degree in the equation of the inverse curve will be
$(k')^2(m-2)(Dx^2 + Exy + Fy^2)$, hence that curve will have a node at the cen-
tre of inversion, and the above equation determines also the tangents to the
inverse curve at that point. Thus in the case of the hyperbola, equ’n (5),
which is a non-circular conic $[m = 2, q = 0]$, with real points at infinity
in the direction $y = \pm x$; the inverse curve (6) is a bi-circular quartic with
a double point at the centre of inversion, at which point the tangents are
$y = \pm x$. When, as in this example, the points at infinity are real and
different the inverse has a crunode. When, as in the case of the parabola,
eq (4), the points at infinity are coincident, the inverse has a cusp at the
centre of inversion like the cissoid of eq. (3). Finally, when the points at
infinity are imaginary, the inverse has imaginary branches passing through
the centre of inversion, which is therefore an acnode.

12. In general, $q$ being the index of circularity, the terms of highest
degree are of the form
$$(Hx^{m-q} + Ix^{m-r-1}y \ldots)(x^2 + y^2)^r$$ the circular points at infinity are multiple points of the order $q$, and there
are $m-q$ other points at infinity. The inverse curve has a multiple point
at the centre of inversion of the order $m-q$. Thus all the points at infini-
ity of the given curve are represented by real or imaginary branches of the
inverse curve passing through the centre, except the $2q$ points which coin-
cide with the circular points at infinity.

13. Conversely, the given curve being a $q$-circular $n$-ic, $m = n - q$.
If the centre of inversion is not a point of the curve, the inverse is an
$m$-circular $2m$-ic. If the centre of inversion is an ordinary point of the curve,
the inverse is an $(m-1)$-circular $(2m-1)$-ic, and has one real point at infinity.
If the centre of inversion is a node, the inverse is an \((m - 2)\)-circular \((2m - 2)\)-ic, having two real, coincident or imaginary points at infinity, according as the node is a crunode, cusp or acnode. In general, if the centre of inversion is a multiple point of the \(s\) order, the inverse is an \((m - s)\)-circular \((2m - s)\)-ic, with \(s\) real or imaginary points at infinity corresponding to the \(s\) branches of the given curve at the centre of inversion.

14. It is evident geometrically that a node or cusp, not at the centre of inversion, inverts into a similar point, so that the possession of these singularities affords a sub-classification within the circular orders such that a curve and its inverse belong to the same sub-class.

Thus a non-singular bi-circular quartic can invert only into non-singular bi-circular quartics or circular cubics. The crunodal bi-circular quartics and circular cubics, with the hyperbolas, form another sub-class the members of which invert only one into another. The cuspidal bi-circular quartics and circular cubics, together with the parabola, form another class of curves bearing the same mutual relations, and finally the acnodal varieties of these higher curves are similarly associated with the ellipse. We have an example in the case of the cissoid of equation (7), which has a cusp at the point \((-a, 0)\); this cusp inverts into a cusp, which, since the modulus employed was \(a^2\), is situated at the same point. Accordingly, if we transfer the origin in eq. (8) to the point \((-a, 0)\), we have the equation

\[ r^4 - arx - 2ay^2 = 0, \]

showing that the curve has a cusp at which the tangent is the axis of \(x\).

If we invert this curve with respect to its present origin, the cusp, we shall have the parabola

\[ a^2 - ax - 2y^2 = 0. \]

Determination of the Locus of \(O\). (See Fig. on p. 22.)

By Christine Ladd, Union Springs, N. Y.

The perpendicular on \(\gamma\) from \(\alpha'\beta'\gamma'\) is

\[ a\beta' - \beta a' = (\beta\gamma' - \gamma\beta')\cos B + (\gamma\alpha' - \alpha\gamma')\cos A, \]

and the line joining its intersection with \(\gamma\) to the vertex \(C\) is

\[ a(\beta' + \gamma'\cos A) - \beta(\alpha' + \gamma'\cos B) = 0. \]

The condition that the three lines of this kind meet in a point is

\[ (\alpha' + \beta'\cos C)(\beta' + \gamma'\cos A)(\gamma' + a'\cos B) \]

\[ = (a'\cos C + \beta')(\beta'\cos A + \gamma')(\gamma'\cos B + a'), \]
or the locus of \( a'\beta'\gamma' \) is
\[
a(\beta^2 - \gamma^2)(\cos A - \cos B \cos C) + \beta(\gamma^2 - \alpha^2)(\cos B - \cos C \cos A) + \gamma(\alpha^2 - \beta^2)(\cos C - \cos A \cos B) = 0. \quad \ldots \ldots (1)
\]

This equation represents a cubic which passes through the vertices of the triangle and cuts the sides \( a, \beta, \gamma \) again at their respective intersections with
\[
\beta(\cos C + \cos A \cos B) = \gamma(\cos B + \cos C \cos A),
\]
\[
\gamma(\cos A + \cos B \cos C) = \alpha(\cos C + \cos A \cos B),
\]
\[
\alpha(\cos B + \cos C \cos A) = \beta(\cos A + \cos B \cos C).
\]

The equation \( (1) \) is satisfied by the coordinates \( 1, 1, 1; \cos A, \cos B, \cos C; \) and \( \cos B \cos C, \cos C \cos A, \cos A \cos B; \) hence the curve passes through the centres of the inscribed and circumscribed circles and through the orthogonal centre of the triangle.

If \( A = B, \) \( (1) \) becomes
\[
(a - \beta)[\alpha(\gamma^2 + \alpha\beta) - a(\alpha\gamma + \beta\gamma)] = 0.
\]
\( a - \beta \) is the bisector of the angle \( C \) and, on applying the criterion (Todd-hunter's Conic Sections, p. 306),
\[
\alpha(\gamma^2 + \alpha\beta) - a(\alpha\beta + \beta\gamma) = 0
\]
is found to be an hyperbola. One branch passes through \( A \) and the middle point of \( AC, \) the other, through \( B \) and the middle point of \( BC. \) Its transverse axis is parallel to \( AB. \) If \( A = B = C, \) \( (1) \) becomes
\[
(a - \beta)(\beta - \gamma)(\gamma - a) = 0,
\]
the bisectors of the angles of the triangle.

---

**Note by Artemas Martin.** — There is an omission in my solution of problem 139, p. 29, Vol. IV. In the first line the coefficients \( 1 + \sigma^2 \) and \( 1 - \sigma^2 \) should be divided by \( \sigma^4. \) In the third line write \( (2\sigma^3 - 1) + \sigma^3 \) for \( (2 - \sigma^3) \) and \( (1 - \sigma^3) \div \sigma^3 \) for \( (1 - \sigma^3). \)

The corrected result is
\[
A = \frac{4\pi a^2}{3\cos^5} \left[ \alpha(1 - \sigma^2)\sqrt{\frac{1}{\sigma^2}} + \left(\frac{2\sigma^3 - 1}{\sigma^3}\right)[F'(\sigma)] + \left(1 - \sigma^3\right)[F'(\sigma)] \right].
\]

**Query, by Prof. A. Hall.** — The approximate value of the definite integral
\[
\int_{\sigma}^{\varphi} \sqrt{\sin \varphi} \, d\varphi
\]
is 1.198\ldots. Is there a convenient way of computing this numerical value?
SOLUTION OF THE GENERAL EQUATION OF THE FIFTH DEGREE.

TRANSLATED BY DR. A. B. NELSON, DANVILLE, KENTUCKY.

(Continued from page 14.)

§. 15.

Let there be given an equation of the 6th degree:

I. \[ x^6 - 3bx^4 + Bx^2 - Cx + Dx^2 - Ex + G = 0. \]

I wish to construct its Capital.

\[ x = (x_i + \frac{1}{6}b), \] thus:

\[ x^6 = x_i^6 + 3bx_i^4 + \frac{1}{4}b^2x_i^2 + \frac{1}{6}b^3x_i + \frac{1}{8}b^4 + \frac{1}{10}b^5x_i + \frac{1}{12}b^6 \]

\[ -3bx^4 = -3bx_i^4 - \frac{1}{2}b^2x_i^2 - \frac{1}{2}b^3x_i - \frac{1}{2}b^4x_i - \frac{1}{2}b^5x_i - \frac{1}{2}b^6 \]

\[ Bx^2 = Bx_i^2 + 2bBx_i^2 + \frac{1}{3}b^3Bx_i^2 + \frac{1}{4}b^4Bx_i + \frac{1}{5}b^5B + \frac{1}{6}b^6B \]

\[ -Cx^3 = -Cx_i^3 - \frac{1}{3}bCx_i^3 - \frac{1}{4}b^2C - \frac{1}{5}b^3C \]

\[ Dx^2 = Dx_i^2 + bDx_i^2 + \frac{1}{2}b^3D \]

\[ -Ex = -Ex_i - \frac{1}{2}bE \]

\[ G = G \]

Therefore:

II. \[ x_i^6 + (\frac{1}{6}b^3 + B)x_i^4 - (5b^3 - 2bB + C)x_i^2 \]

\[ + (\frac{1}{6}b^3 + \frac{1}{2}b^2B - \frac{1}{3}b^3C - \frac{1}{4}b^4D)x_i \]

\[ - (\frac{1}{6}b^3 + \frac{1}{2}b^2B - \frac{1}{3}b^3C - \frac{1}{4}b^4D + \frac{1}{5}bE + \frac{1}{6}b^6) = 0. \]

§. 16.

In §. 10 I put \((yv + yt + vz + zu + ut) = m\) and found in §. 11 that there are 6 symmetrical halves of \(b\) whose sum = 3b, viz.:

I. \((yv + yt + vz + zu + ut)\)

\((yz + yu + vz + vt + ut)\)

\((vy + yu + vt + zu + x)\)

\((yz + yt + vu + vt + zu)\)

\((vy + yz + vu + x + ut)\)

\((yv + yt + vz + vu + x)\).

I use these six expressions as the unknown quantities of an equation of the 6th degree and construct the latter, viz.:
II. $x^8 - Ax^6 + Bx^4 - Cx^2 + Dx - Ex + G = 0$,
which gives:

III. $A = 3b,
\[ B = (3b^2 - 5d), \]
\[ C = (b^3 - 10bd), \]
\[ D = (-8b^3d + b^5e - 25b + 15d^2), \]
\[ E = (-3b^3d + b^5e - 25bce + 15bd^2 + \sqrt{P}), \]
\[ G = \frac{-15b^3ce - 25bd^2 + 125b^3d - 78b^5d - 500ce - 2d^4 + 10d^5 + b\sqrt{P}}{2}. \]

If I now construct the capital of equation II., I find by the use of §. 15,

IV. $(-\frac{1}{4}b^2 + B) = \frac{1}{4}(-15b^3 + 12b^3d - 12d) = -\frac{1}{4}(3b^2 + 20d)$
\[ (5b^2 + 2bB + C) = (5b^2 - 6b^3 + 10d + 2b - 10d) = 0, \]
\[ (\frac{1}{4}b^2 + B) = (\frac{1}{4}b^2 + \frac{1}{2}b^2d - \frac{3}{2}b^2c - 25b + 15d^2) = \frac{1}{4}(3b^2 - 8b^2d + 16b^2e - 400ce + 240d^2), \]
\[ (\frac{1}{2}b^2 \cdot \frac{1}{2}b^2b + \frac{1}{2}b^2c - b^2d + E) = (\frac{1}{2}b^2 + \frac{3}{2}b^2 + \frac{1}{2}b^2d + \frac{1}{2}b^2c - 8b^2d - b^2c + 25bce - 15bd^2 - 3b^2d + b^2e - 25bce + 15bd^2 + \sqrt{P}) \]
\[ = \sqrt{P}. \]
\[ (-\frac{1}{8}b^2 + b^2 + b^2 + B + \frac{1}{2}b^2C + \frac{1}{2}b^2D - \frac{1}{2}bE + G) = \left[ -\frac{1}{8}b^2 + \frac{1}{8}(3b^2 - 5b^2d) - \frac{1}{8}(b^2 - 10b^d) + \frac{1}{8}(-8b^3d + b^5e - 25b^3c + 15b^3d^2) + \frac{1}{8}(3b^2d - b^5c + 25b^2c - 15b^2d) + \sqrt{P} \right] \]
\[ = \frac{1}{8}(b^2 - 28b^3d - 16b^5e - 80b^5c - 176b^6d^3 + 400b^6c^2 + 224b^6d^2 - 1600b^6d - 64b^6e + 320b^6d). \]

Consequently

V. $x^8 - \frac{1}{4}(3b^2 + 20d)x^6 + \frac{1}{4}(3b^2 - 8b^3d + 16b^2e - 400ce + 240d^2)x^4 - \sqrt{P}x^2 + \frac{1}{8}(b^2 + 28b^3d - 16b^5e - 80b^5c - 176b^6d - 4000b^6c^2 + 224b^6d^2 - 1600b^6d - 64b^6e + 320b^6d) = 0.$

Whence it follows next that an equation of the 5th degree which has two equal unknown quantities is solvable algebraically; that is to say its unknown quantities can be represented by an algebraic function of $b, c, d, e,$ because then the product of the differences of the unknown quantities has a factor $(y - y) = 0,$ and is thus itself $= 0.$ Therefore equation V can be resolved into a cubic with unknown quantities of quadratic form; consequently $x,$ is found equal to $m$ and hence the 5 unknown quantities of the given equation, $a^2 + bx^2 - cx^2 + dx - e = 0,$ are known.

*$_\sqrt{P}$ = the product of the 10 differences in one direction. See §. 4.
But in the foregoing general form also the solution of equation \( V \) is feasible. This is indeed an equation with its highest exponent 6, but in proper significance not an equation of the 6th degree* whose general solution formula is therefore not applicable here, since this (for the case in which the absolute term \( = 0 \)) must embrace in itself the possibility of a conversion into the formula for equations of the 5th degree, and therefore must contain radicals whose index is 5. The formula for \( m \) cannot have such radicals. In § 8 it was shown that outside the outer radicals of the 5th degree, others could not generally occur in the irrational quantity of the formula of the 5th degree.

In its generality \( m \) indeed has 12 dimensions, but by combining the only quadratic expression \( \sqrt{P} \) (see p. 1, preliminary remark) occurring among the coefficients, the irrational quantities involved will have only 6 dimensions, and only radicals with the indices 2 and 3 can appear in the same.

I assume that if in equation \( V \) \( t \) and consequently \( e \) become \( = 0 \) (thus \( V \), the fundamental equation of the 4th degree) all the elements which have the factor \( e \), disappear.

I write equation \( V \)

\[
\begin{align*}
(1) \quad x_r^6 - \frac{1}{4} A x_r^4 + \frac{1}{16} B x_r^2 - D x_r + \frac{1}{4} C & = 0. \\
(2) \quad x_r^6 - \frac{1}{4} a x_r^4 + \frac{1}{16} b x_r^2 - \delta x_r + \frac{1}{4} \gamma & = 0,
\end{align*}
\]

and I have for this case:

VII. \( a = (3b_r^2 + 20d_r) \),

\( \beta = (3b_r^2 - 8b_r d_r + 16b_r c_r + 240d_r^2) \),

\( \gamma = (-6b_r^2 - 28b_r d_r - 16b_r c_r - 176b_r d_r + 224b_r c_r d_r - 64c_r^2 - 320d_r^2) \),

\( \delta = d_r \sqrt{(16b_r^2 + 144b_r c_r + 128b_r d_r - 4b_r c_r^2 - 27c_r^2 + 256d_r^2)} \),

which last expression I will also write \( d_r \sqrt{P(4)} \), for the expression under the radical is the product of differences for the equation of the 4th degree.

From the foregoing equations (1) to (3) I am now able to represent the three coefficients \( b_r, c_r \), and \( d_r \) in terms of \( a, \beta, \gamma \), and indeed by means of an equation of the 4th degree, viz., for \( d_r \):

VIII. \( \left( \frac{a - 20d}{3} \right)^4 = \frac{520}{352} a^3 (a - 20d) + \frac{170}{352} b_r a^2 (a - 20d) - \frac{82}{352} c_r a (a - 20d) + \frac{352}{82} \beta (a - 20d) = 0; \)

thus \( d_r = \text{function } a \beta \gamma. \)

*It has properties of the equation of the 2nd and of the equation of the 3rd degree.
(3) \( \delta = d_\gamma^2 P(4) \) signifies the determined relation of \( b, c, d \) to one another. If I introduce into (3) the values of \( b, c, d \), found from (1), I have:

\[
(4) \left( \frac{a-20d}{3} \right)^4 - \left( \frac{404472a^2+575280\beta}{340704a} \right) \left( \frac{a-20d}{3} \right)^3 + \left( \frac{171986a^2+540930a\beta-453600\gamma}{340704a} \right) \left( \frac{a-20d}{3} \right)^2 - \left( \frac{27583a^4+217910a^2\beta-596700a\gamma}{340704a} \right) \left( \frac{a-20d}{3} \right) + \left( \frac{859a^8+33840a^6-73575a^4\beta^2-148500a\gamma+140800000}{340704a} \right) = 0,
\]

or

(5) \( \delta = \text{function } a\beta\gamma \). And by equating (4) and (1) I moreover depress the degree of the equation for \( d \).

I assume that the values \( A, B, C, D \) [VI. (1)] may be introduced into the equation [from (5)].

IX. (1) Function \( a\beta\gamma \delta = 0 \); thus

(2) Function \( A, B, C, D = N \) having been formed, \( N \text{ must } = 0 \).

For if \( N \) had any value it must contain the factor \( e \), since

Function \( A, B, C, D = N \) immediately becomes

Function \( \alpha \beta \gamma \delta = 0 \), when I put \( e = 0 \).

But if \( N \) contain the factor \( e \), then the other member of the equation must have an equal factor. Hence there would arise a new equation;

(3) Function \( b, c, d, e = e \). Whence I would be able to deduce

(4) Function \( b, c, d = e \); but this is contradictory to the character of the equation:

\[ a^2 + bx^2 - cx^2 + dx - e = 0, \]

which is general: one whose coefficients may be chosen arbitrarily and therefore must be independent of one another. From

(5) Function \( A, B, C = D \) it follows that the unknown quantities of equation VI. (1), answering to V., may be represented by a function of \( A, B, C \).

§. 17.

Let

I. \( a^4 + bx^2 - cx + d = 0 \).

The 6 symmetrical halves of \( b = yv + yz + yu + vz + vu + xu \), whose sum = \( 3b \), are:

II. \( (yz + vz + xz) = n \), and \( n = (n - \frac{1}{2}b) \)

\( (yz + yu + vz) \)
(yz+uy+zu)
(yz+vu+zu)
(yz+uv+zu)
(yz+u+v+zu)

If I represent \( n \) (which for equations of the 4th degree is supposed to be known) by means of the coefficients \( b, c, d \), provided

\[
f = \sqrt{\frac{1}{2} \left( 2b^8 - 72bd + 27d^2 \right) + \sqrt{\left( 2b^8 - 72bd + 27d^2 \right)^2 - (b^2 + 12d)^8} },
\]

\[f' = \frac{1}{2}f(-1+\sqrt{-3}), \quad f'' = \frac{1}{2}f(-1-\sqrt{-3}),\]

I have

\[
III. n = \frac{2(f + \frac{b^2 + 12d}{f}) + (f' + \frac{b^2 + 12d}{f'}) + \sqrt{-12d + 3(b + f' + \frac{b^2 + 12d}{f'})}}{6}.
\]

I now assume farther that from equation § 16. VI. (1)

IV. \( m = (m - \frac{1}{2}b) = \) function \( ABC \) (without \( D \)) may be discovered, which is possible, since \( m \), is a function of the coefficients of its equation and \( D \) is a function of \( ABC \); thus if I put \( e = 0 \), IV will be at once converted into III. But as it is impossible that any further change can occur in the position of the radicals in the formula for \( m \), than that one of them shall entirely vanish or an indicated root appear as actually accomplished, formula III. is in reality a symbol of the expression for \( m \).

If I find from §. 16. VIII. the values of \( b, c, d \), according to \( a, \beta, \gamma \) and introduce these results into III., there thus arises another formula for \( n \), which I designate

V. \( n = \) Function \( \alpha \beta \gamma \): similarly I am able to represent the 5 other values of \( n \).

If I now construct from these 6 values of the unknown quantities, equation §. 16. VI. (2) with respect to its coefficients, \( \alpha \beta \gamma \) will occupy the same position in the result which they now have in formula §. 16. VI. (2).

If I therefore exchange in formula V. \( \alpha \) with \( A \), \( \beta \) with \( B \), and finally \( \gamma \) with \( C \), there arise 6 expressions \( m \), etc., which, treated as above described for \( n \), and its correspondents must give the formula VI. (1) (with previous exclusion of \( D \)).

But, moreover, the coefficient \( D \) must appear in the position which in formula §. 16. VI. (2) the coefficient \( \delta \) occupies; for \( \delta \) was derived from \( D \) by putting \( e = 0 \) in the equivalent function \( A \cdot B \cdot C \). Thus, in the next place, all the terms in \( \delta \) and \( D \) which do not have the element \( e \), must be identical.

And it follows from this mode of derivation also that the position of the extra elements in \( D \) and \( \delta \) is generally the same, for by having taken the
above described collocation of the coefficients $\delta$ from $n$, and its 5 corresponding values, I completed the operation indicated in the equation $\delta = a \beta \gamma$.

If I now put together $D$ and $m$, etc., the extra symbols in $A$, $B$, $C$ must pass through the same algebraic processes of transformation as was necessary with $n$, etc. in its formation from $\delta$, and the result must consequently be $D$.

§. 18.

From the investigations in §§. 16 and 17 the possibility of the algebraic representation of $m = m - \frac{1}{2}b$ results, as follows:

From the equation §. 16. VIII. (which, as already mentioned, can be depressed, and indeed to the first degree for $d$) I find the values of $b$, $c$, $d$ in terms of $\alpha$, $\beta$, $\gamma$ [§. 16. VI. (2)], exchange them with $A$, $B$, and $C$ respectively, and introduce the result thus found into the formula §. 17. III. Hence $m$, becomes known.

From $m$, I have $m = (m, \frac{-1}{2}b)$, and hence I am able (§. 14.) to develop $y$, with which the solution of the equation of the 5th degree is complete.

I had indeed striven earnestly not to go beyond the solution of the equation of the 5th degree, but my material crowds its limits, and one observation breaks over the rule.

The direct proof that the exchanges of the coefficients of VI. (1), made in the formula for the unknown quantities of VI. (2), is well grounded, is connected with the basis of the problem of the general solution, viz., to establish the relation between $(n-1)$ constants (§. 5) of an equation of the $n$th degree and the $(n-1)$ coefficients of a complete equation of the $(n-1)$th degree.

There are thus in the present case the four coefficients of the complete equation of the 4th degree to be represented in the radicals of formula III., for it is possible also, without reference to particular properties of the products of differences, to construct from these 4 elements and the 4 coefficients of equation VI. (1) four equations of the 4th degree for determining $a$, $b$, $c$, and $d$, and consequently, to discover the coefficients.

[The translator of the foregoing solution is conscious of having executed his task imperfectly. There are two or three expressions especially which, either from the difficulty of the thought or the obscurity of its expression in the original, he fears he has not adequately rendered. It is hoped, however, that in the solution as given the reasoning as a whole will be intelligible to all.—Translator.]

*Or: Let the product of differences for the equation of the 5th degree be $P(5) = $ function $A$, $B$, $C$. By putting $c = 0$ I exchange $A$ with $a$, $B$ with $\beta$, $C$ with $\gamma$, and there arises $P(4) = $ function $a$, $\beta$, $\gamma$. Therefore, if in the last expression for $a$, $\beta$, $\gamma$ I substitute $A$, $B$, $C$ respectively, $P(5)$ is reproduced.
ON PROBLEM 132.

BY PROF. A. HALL, WASHINGTON, D. C.

The integral required in this question is a special case of the one discussed by Legendre in his treatise on elliptic functions, Chap. XXVI., p. 165. Legendre shows that the elliptic integrals disappear, and that the result can be expressed in logarithmic and circular functions. His method is complicated however, and Clausen has obtained the result by the following simple substitutions. Let

$$z = \frac{x - 1}{\sqrt[3]{x^3 - 1}}; \quad z' = \sqrt[3]{x^3 - 1}; \quad z'' = \frac{(x - 1)^2}{\sqrt[3]{x^3 - 1}};$$

Then

$$dz = -\frac{x^3 - 3x^2 + 2}{2\sqrt[3]{x^3 - 1}} \, dx; \quad dz' = \frac{3}{2} \frac{x^2 \, dx}{\sqrt[3]{x^3 - 1}};$$

$$dz'' = \frac{1}{2} \frac{x^4 + 2x^3 - 3x^2 + 4x + 4}{\sqrt[3]{x^3 - 1}} \, dx.$$  

Hence

$$dz = \frac{1}{1 - 3z^2} \frac{x^3 - 2x - 2}{x^3 - 2x + 4} \frac{dx}{\sqrt[3]{x^3 - 1}}; \quad dz' = \frac{3}{2} \frac{x^2 \, dx}{x^3 + 9};$$

$$dz'' = \frac{1}{2} \frac{(x - 1)(x^3 + 3x^2 - 4)}{2(x + 2)(x^3 + 3x^2 - 4) \sqrt[3]{x^3 - 1}} \frac{dx}{\sqrt[3]{x^3 - 1}};$$

The integrations can now be performed and we have

$$\int \frac{xdx}{(x^3 + 8)\sqrt[3]{x^3 - 1}} = \frac{1}{12\sqrt[3]{3}} \log \left( \frac{1 + z\sqrt[3]{3}}{1 - z\sqrt[3]{3}} \right) + \frac{1}{18} \tan^{-1} \frac{z}{3} + \frac{1}{18} \tan^{-1} \frac{z''}{3}$$

$$= \frac{1}{12\sqrt[3]{3}} \frac{1}{\sqrt[3]{x^3 + x + 1} + \sqrt[3]{x - 1}} \frac{3}{3} + \frac{1}{18} \tan^{-1} \frac{3(x - 1)}{4(x^3 - 1)^{3/2}}.$$

REMARKS BY PROF. ORSON PRATT, SEN., SALT LAKE CITY, UTAH.

—the six propositions, published in the Analyst, Vol. III, No. 6, and demonstrated in the last No., are not “presented as the basis of a theory of gravitation.” They have no bearing on or reference to the cause of the force called gravity, or to the exploded hypothesis of an ethereal pressure, exerted in the direction of a gravitating center.
The only modification that I have given to the Newtonian theory, is the assumption that the force is not transmitted through space instantaneously, but with the velocity of light and all other solar radiations.

On the assumption that there is no resisting medium, La Place has demonstrated that the stability of the system depends on the almost instantaneous transmission of the force.

My theory is that the great ethereal ocean is a resisting medium, and that the stability of the universe must, consequently, depend upon an orbital force constantly acting, and that the gravitating force, transmitted with any given velocity must necessarily give rise to such orbital force.

In the second proposition, I have demonstrated the only law of ethereal density which can possibly render the system stable.

Ethereal substance, like all other matter, is subject to gravitation; therefore it must have "a density increasing toward the centers of bodies immersed in it." For the same cause, certain dimensions of these ethereal atmospheres or envelopes, must accompany bodies in their axial rotations.

The laws which I have demonstrated in regard to the constantly acting orbital force, are the necessary results of a still more general and universal law, which may be expressed as follows:

Every Particle of Matter in the Universe transmits its Force to every other Particle, with the velocity of light, by virtue of which every Moving Particle exerts upon every other Particle an Aberrating Force at right angles to the connecting line, which is directly as the Mass, multiplied into the Velocity of Aberration, and inversely as the Square of the Distance from each.

[The foregoing Remarks by Prof. Pratt furnish a pertinent and sufficient reply to the suggestion we made at page 19. We embrace this opportunity to say, however, that we dissent entirely from the theory outlined by Prof. Pratt in the above remarks and in the propositions referred to.

We accept the existence of an ether, but such an ether as the undulatory theory of light requires, and, as defined by Sir John Herschel, "having inertia but not gravity." We deny that there is any evidence of the progressive motion of gravity, unless the force should turn out to be a result of ethereal motion which Prof. Pratt denies. We also deny that the system of the world is necessarily, or even probably, permanent, because both observation and analogy teach, we think, that systems, like individuals, have a beginning, attain their maturity, and then decay and die; the physical universe undergoing, as we believe, an infinite series of changes through every grade of organization from that of an individual plant or animal to that of a solar, or a stellar system.—Ed.]
SOLUTION OF PROBLEMS IN NUMBER ONE.

Solutions of problems in No. 1 have been received as follows:
From Marcus Baker, 141, 142, 143 and 148; G. M. Day, 141, 142 and 143; Prof. H. T. Eddy, 149; Henry Gunder, 141, 142, 143, 144 and 148; Henry Heaton, 147 and 150; Prof. A. Hall, 149; Wm. Hoover, 143, 146 and 148; Christine Ladd, 141 and 142; W. L. Marcy, 141, 142, 143, 144, 148 and 150, also I to V inclusive, of Prof. Pratt's problems, (p. 186, Vol. III); Artemas Martin, 141, 146, 148 and 149; D. J. Mc. Adam, 143, 146 and 148; W. V. Mc. Knight, 144; J. Scheffer, 141, 142, 143 and 148; E. B. Seitz, 141, 142, 143, 144, 145, 146, 147, 148 and 149.

141. "Let $ABC$ be any plane triangle, $R$ the radius of its circumscribing circle, $r'$ the radius of the escribed circle opposite $A$, $p_1$, $p_2$, $p_3$ the perpendiculars from the center of the circumscribing circle upon $a$, $b$ and $c$. Show that $r' = R = p_2 + p_3 - p_1$.

SOLUTION BY W. L. MARCY, DEL NORTE, COLORADO.

The relation [Chauvenet, Trig., p. 78, (298)]

$$r + R = - \cos A + \cos B + \cos C + 1$$

Gives

$$r - R = R(\cos C + \cos B - \cos A).$$

But

$$R \cos C = p_3; \ R \cos B = p_2; \ R \cos A = p_1.$$

Substituting, there results

$$r - R = p_3 + p_2 - p_1.$$

142. "In a plane triangle $ABC$, $AO'$ is drawn bisecting the angle $A$; from $O'$, the center of the escribed circle, a perpendicular to $AO'$ is drawn, meeting $AB$ produced in $T'$, and from $T'$ a perpendicular to $AB$ is drawn, meeting $AO'$ produced in $O''$; with $O''$ as center and $O''T'$ as radius a circle is described; prove that this circle is tangent to the circle circumscribed about the triangle $ABC$.

SOLUTION BY G. M. DAY, LOCKPORT, N. Y.

Put $O''T' = R'$, radius of circumscribed circle $= R$, and let its center be denoted by $O$. Let $OO'' = d$, and $r'$ = radius of escribed circle. Then is

$$d^2 = R^2 + R' \csc^2 \frac{A}{2} - 2R' \csc \frac{A}{2} \cos \frac{A}{2}(B - C). \ldots (1)$$
If the circles whose centres are \(O\) and \(O''\) are tangent, we must have
\[ d^2 = (R + R')^2 \ldots \ldots (2), \quad R' = r' \sec \frac{1}{2} A \ldots \ldots (3) \]
Combining (1), (2) and (3) we obtain
\[ \frac{r'}{2R \sin^2 \frac{1}{2} A} = 1 + \frac{\cos \frac{1}{2} (B - C)}{\sin \frac{1}{2} A} \ldots \ldots \ldots (4) \]
But each member of (4) = 2 \(\cos \frac{1}{2} B \cos \frac{1}{2} C + \sin \frac{1}{2} A\), 
the proposition is proved.

143. "Required the area of the triangle formed by the tangent to the curve whose equation is \(x^r + y^r = r^r\) and the axes of \(x y\)."

SOLUTION BY HENRY GUNDER, NORTH MANCHESTER, IND.

Let \(a\) and \(b\) be the coordinates of the curve, and \(x\) and \(y\), of the tangent line.

The equation to the tangent is
\[ y - b = \frac{db}{da} (x - a) \ldots (1). \]
From \(a'^r + b'^r = r'^r\), \(\frac{db}{da} = -\frac{b'^r}{a'^r}\ldots (2)\)
Therefore, by substitution, (1) becomes \(y - b = \frac{b'^r}{a'^r} (a - x)\ldots (3)\)
Making \(x\) and \(y\) each = 0, in succession, in (3) we get,
\[ y = a'^r x \ldots (4), \quad x = b'^r x \ldots \ldots \ldots \ldots (5) \]
Hence the area = \(\frac{1}{2} \phi'(ab'^r) = \frac{1}{2} r\phi'(abr)\).

144. "A cylindrical post, diameter \(d\), stands perpendicularly on a level plane with a rope, diameter \(d'\) and length \(l\), attached to the bottom of the post and lying stretched on the plane. How far will a person walk who takes hold of the end of the rope and winds it around the post by walking around it always keeping the rope parallel to the plane and also keeping the coils on the post at a distance of 3\(d'\) from each other?"

SOLUTION BY W. V. MC. KNIGHT, PARKSVILLE, KENTUCKY.

Let \(BOM\) represent the semi-base of the post, \(A\), its center, \(OQ\), a part of the rope, unwound, and \(MQ\) the projection upon the plane, of the curve described by the end of the rope in being wound to \(M\). Because one spire of the rope about the cylinder is greater than the circumference of the cylinder, the radius of curvature of any point of the required curve is greater than the radius of curvature of the corresponding point of the arc of an involute to the base of the post; let \(v (= PQ)\) represent this difference.
Draw $OL$ and $OC$ perpendicular to the axis of $X$, and $QN$ perpendicular to $OL$. We have $AO = \frac{1}{2}(d + d')$, which put $= a$, and let $AC = x$, $QC = y$, curve $MQ = z$ and angle $OAM = \varphi$. Let $s$ represent the difference between the length of one spire and the circumference of the post; then is $2\pi a\varphi : s : v$; Therefore $v = s\varphi + 2\pi$, and $OQ = a\varphi + s\varphi ÷ 2\pi = (a + s ÷ 2\pi)\varphi = h\varphi$ = the radius of curvature for the point $Q$. We have

\[
AL + NQ = x = a \cos \varphi + h\varphi \sin \varphi, \quad \cdots \cdots \cdots \quad (1)
\]

\[
OL - NO = y = a \sin \varphi - h\varphi \cos \varphi. \quad \cdots \cdots \cdots \quad (2)
\]

Differentiating (1) and (2), squaring, and adding them together, observing that $ds^2 = dx^2 + dy^2$, we have, putting $h = a = k$,

\[
ds = \sqrt{(k^2 \varphi^2 + k^2)}d\varphi. \quad \cdots \cdots \cdots \quad (3)
\]

Integrating (3) we find

\[
z = \frac{1}{2}\varphi \sqrt{(k^2 \varphi^2 + k^2)} + (k^2 + 2k)\log[\sqrt{(k^2 \varphi^2 + k^2)} + h\varphi] + C.
\]

When $z = 0 \varphi = 0; \cdots C = (k^2 + 2k)\log k$; and when $h\varphi = l, \varphi = l ÷ h$ = $c$ say. Substituting these values for $C$ and $\varphi$, we find

\[
z = \frac{1}{2}c \sqrt{(k^2 \varphi^2 + k^2)} + (k^2 + 2k)\log \{ \sqrt{(k^2 \varphi^2 + k^2)} + h\varphi \} ÷ k,
\]

which is the distance walked by the man in winding the rope about the post as required, provided he is able to keep the rope horizontal while walking on a level plane.

[As $z$ is the projection upon the plane of the curve described by the end of the rope, if the person is supposed to ascend as the end of the rope ascends, in winding around the post, the distance walked, in that case is a curve of double curvature and is represented by the projection of $z$ upon the surface of an hyperboloid.—In all the other solutions, of this question, that have been received, the curve found is the involute of a circle whose circumference is the length of one spire around the post, and is therefore a slightly different curve from the one required. As the distance between the spires, if taken perpendicular to the rope, and if in general measurable on the surface of the post, is a curve, we infer that by “a distance of $3d'$ from each other” is meant a vertical distance of $3d'$. On this supposition the value of $h$, as determined by Mr Seitz, is

\[
\frac{\pi(d + d')}{{\pi(d + d')} \sqrt{[\pi^2(d + d') + 8d'^2]} - d'^2}.
\]

Ed.]

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145. "The product of the lengths of tangents from the radical center of three circles on any pair of circles through the intersections of the given circles is equal to the product of the lengths of tangents from the same point on any pair of circles tangent to the given circles."

**SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.**

Let $O$, $O'$, $O''$ be the three given circles, $O$, $O'$ intersecting in $a$, $b$, and $O'$, $O''$ in $c$, $d$, and let $M$, $N$ be a pair of circles tangent to the three given circles.

Produce $a$ $b$ and $c$ $d$ till they meet in $V$, which is the radical centre of $O$, $O'$, $O''$. Since the circles $O$ and $O'$ touch the circles $M$ and $N$, and the contacts are of the same kind, the radical axis of $O$ and $O'$ passes through the external center of similitude of $M$ and $N$; and for the same reason the radical axis of $O'$ and $O''$ passes through the same point. Therefore, $V$ is the external center of similitude of $M$ and $N$.

Let $T$ denote the length of the tangent from $V$ to any circle through $a$, $b$ or $c$, $d$, and $T'$, $T''$ the lengths of the tangents from $V$ to the circles $M$ and $N$. Then by well known principles we have $T^2 = V.a.V.b$, and $T'.T'' = V.m.V.n = V.a.V.b$; $\therefore T'^2 = T'.T''$.

146. "Given a semicircle and a circle, place the latter so that it will cut the former; what is the probability that its center will fall within the former?"

**SOLUTION BY ARTEMAS MARTIN, ERIE, PA.**

Let $A B C$ be the given semicircle, radius $R$. With same center $O$ describe a semicircle $DEF$, radius $R+r$, $r$ being the radius of the given circle. With centers $A$ and $B$, and radii $r$, describe the quadrants $DG$ and $EH$.

If the center of the circle be placed within the boundary $DFEHG$ it will cut the semicircle.

Area $ABC = \frac{1}{2} \pi R^2$, area $DEF = \frac{1}{2} \pi (R + r)^2$, area $AGBH = 2Rr$, and
area $ADG = area BEH = \frac{1}{2} \pi r^2$. Hence
\[
p = \frac{\frac{1}{2} \pi R^2}{\frac{1}{2} \pi r^2 + \frac{1}{2} \pi (R+r)^2 + 2Rr} = \frac{\pi R^2}{\pi (R^2 + 2Rr + 2r^2) + 4Rr}.
\]
If $R = r$, $p = \pi + (5\pi + 4)$.

147. "Two rods of equal length have their middle points connected by a string of half the length of one of the rods. If they be thrown on a level floor, what is the chance of their crossing?"

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

For each position that one of the rods may take, the center of the other may fall anywhere on the surface of the circle described on the first as diameter, and the second may make any angle from 0 to $\frac{1}{2} \pi$ with the first. Hence we may regard one of the rods fixed in position on the floor, and consider all the positions the other may take.

Let $AB$ represent the first rod, $C$ its center, $AGHBL$ the circle on $AB$, and let $\theta$ = the angle which the second rod makes with the first. Draw $AD$ and $AF$ each equal to $AC$, making $\angle DAB < \frac{1}{2} \pi$, $\angle FAB > \frac{1}{2} \pi$; draw $DE$ and $FH$ parallel to $AB$, and join $CE$ and $CH$.

When $\angle DAB = \theta$, the second rod will cross the first if its center falls anywhere on the surface $ADEB$; and when $\angle FAB = \theta$, the second rod will cross the first, if its center falls anywhere on the surface $AKGHB$. Put $\Delta C = 1$, $\angle DAB$, or $\Delta FAB = \theta$,

area $ADEB = u_1$, area $AKGHB = u_2$. Then \( u_1 = \frac{1}{2} \theta + \sin \theta \), and

\[ u_2 = 2\theta - \frac{1}{2} \pi + 2 \sin \theta \cos \theta. \]

Hence, doubling, since the center of the second rod may fall on either side of $AB$, we find for the chance required.

\[
p = \frac{2 \int_{0}^{\frac{1}{2} \pi} u_1 d\theta + 2 \int_{\frac{1}{2} \pi}^{\pi} u_2 d\theta}{\int_{0}^{\pi} \pi d\theta} = \frac{4}{\pi^2} \int_{0}^{\frac{1}{2} \pi} (\frac{1}{2} \theta + \sin \theta) d\theta + \frac{4}{\pi^2} \int_{\frac{1}{2} \pi}^{\pi} (2\theta - \frac{1}{2} \pi + 2 \sin \theta \cos \theta) d\theta = \frac{1}{3} + \frac{3}{\pi^2}.
\]
148. "A tortoise, whose shell is circular, radius \( a \), is moving in a straight line at the uniform speed of \( m \) feet per minute, and a fly is running around on the edge of its shell at the uniform rate of \( n \) feet per minute.

Required the equation to the curve the fly describes in space."

**SOLUTION BY D. J. MC. ADAM, WASHINGTON, PA.**

Take the line of motion of the tortoise as axis of \( x \). Let the fly be initially in the axis. At the time \( t \) suppose a line drawn from the fly to the centre of the tortoise to make an angle \( I \) with the axis of \( x \). Then

\[ y = a \sin I \ldots (1), \quad x = mt + a - a \cos I \ldots (2), \quad nt = aI \ldots (3) \]

Eliminating \( t \) and \( I \) in (2), and putting \( m + n = r \),

\[ x = raI + a - a \sqrt{(1 - \sin^2 I)} = ra \sin^{-1}(y/a) + a - \sqrt{a^2 - y^2}. \]

The curve belongs to the family of cycloids. If \( m = n \) it is the common cycloid.

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149. "Prove that

\[ \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{(2ax - x^2)}} = \frac{2}{3a} F\left(c, \frac{\pi}{2}\right), \text{ where } c = \frac{1}{2}.\]

**SOLUTION BY PROF. H. T. EDDY, CINCINNATI, OHIO.**

Let

\[ \frac{x}{a} = \frac{\tan^2 \varphi}{1 + \sec^2 \varphi}, \quad \cdot \cdot \cdot \quad u = \frac{1}{a} \int_0^{\tan^{-1} \frac{1}{2}} \frac{d\varphi}{\sqrt{(1 - \frac{1}{2} \sin^2 \varphi)}}. \]

Again, let

\[ \frac{\sin 2\varphi}{c_1 + \cos 2\varphi} = \frac{2 \tan \varphi}{1 + c_1 - (1 - c_1) \tan^2 \varphi} = \tan \theta, \text{ where } c_1 = \frac{1}{4}, \]

\[ \cdot \cdot \cdot \quad u = \frac{2}{3a} \int_0^{\pi} \frac{d\theta}{\sqrt{(1 - \frac{1}{2} \sin^2 \theta)}} = \frac{2}{3a} F\left(c, \frac{\pi}{2}\right), \text{ where } c = \frac{1}{2}. \]

**SOLUTION BY E. B. SEITZ.**

Let \( x = \frac{1}{2} a (1 - \sin \theta) \). Then \( 2ax - x^2 = \frac{1}{2} a^2 (1 - \sin \theta)(3 + \sin \theta) \), \( a^2 - x^2 = \frac{1}{3} a^2 (1 + \sin \theta)(3 - \sin \theta) \), \( dx = - \frac{1}{2} a \cos \theta \, d\theta \);

\[ \cdot \cdot \cdot \quad \int_0^{\pi} \frac{dx}{\sqrt{(2ax - x^2)}} = \frac{2}{3a} \int_0^{\pi} \frac{d\theta}{\sqrt{1 - \frac{1}{2} \sin^2 \theta}} = \frac{2}{3a} F\left(c, \frac{\pi}{2}\right). \]

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150. "An underwriter insures three vessels, the first an iron steamer, the second a steamer not of iron and the third a sailing vessel, at $20,000, $15,000, and $10,000, respectively. One of them is known to have been
burned at sea; and three persons, A, B, C, whose respective veracities are $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$, report as follows: A, that the lost vessel was an iron steamer; B, that it was not a sailing vessel; and C, that it was a sailing vessel. Required the expectation of loss to the underwriter, the a priori probability of destruction by fire being twice as great in case of a steamer as of a sailing vessel."

SOLUTION BY HENRY HEATON, B. S., DES MOINES, IOWA.

Before receiving the testimony of either A, B or C the chances of burning of the different vessels were as 2, 2, and 1; after A's testimony they were as 6, 2, and 1; after B's, as 24, 8, and 1, and after C's, as 24, 8, and 5, taking them in the order in which they are named in the problem.

Hence the chances of the burning of the different vessels are $\frac{2}{6}$, $\frac{4}{14}$, and $\frac{8}{14}$; and the expectation of loss is $\frac{1}{4}$ of $20000 + \frac{1}{7}$ of $15000 + \frac{1}{8}$ of $10000 = 17567.56\frac{1}{4}$.

PROBLEMS.

151. BY A. W. MASON, CEDAR FALLS, IOWA. — What is the altitude of the maximum cylinder which can be inscribed in a given paraboloid?

152. BY G. M. DAY, LOCKPORT, N. Y. — Find the surface of a right conoid with a circular base.

153. BY J. B. MOTT, NEO., MO. — Prove that if $(1-0\times\frac{1}{2}x\sqrt{-1})^{1+0} - (1-0\times\frac{4}{3}x\sqrt{-1})^{1+0} = \sqrt{-1} ... (1), \ x = [(2\sqrt{-1}+0)][(1+\sqrt{-1})^0 - (1-\sqrt{-1})^0] ... (2); and find the real approximate value of $x$.

154. BY PROF. O. PRATT, SEN., SALT LAKE CITY, UTAH. — Find a general logarithmic theorem for the differentiation of

$$u = y^{x_1 \cdots x_n}, \ z, x_1, x_2, \&c.,$$

being any functions of one variable as $x$.

155. BY PROF. C. BANCROFT, HIRAM, OHIO. — To find the least distance (in miles on the earth's surface) between two places given by latitude and longitude, taking into account the polar compression.

156. BY PROF. JOHNSON. — In a determinant of the nth degree the elements of the principal diagonal consist of units, and of the remaining elements those in the first column are each equal to $a$, those in the second column each equal to $b$ and so on. Evaluate the determinant.
157. By Christine Ladd. — What is the entire number of double points which can be assumed arbitrarily on a curve of the nth degree?

158. By R. J. Adcock. — Let two concentric and similarly placed ellipses, infinitely near each other, be described, the semi-axes of the inner being a and b, and those of the outer a + da, and b + db; show that the minimum distance between their perimeters = $2y/(ab)da+ (a+b)$.

159. By Artemas Martin. — The first of two casks contains a gallons of wine and b gallons of water, and the second contains c gallons of wine and d gallons of water. e gallons are taken from the first and poured into the second cask, and then e gallons are taken from the second cask and poured into the first.

Required the quantity of wine in the second cask after n such operations as the one described above.

160. By Prof. A. Hall. — P and Q being functions of x find the conditions that the equation $ydy + (P - Qy)dx = 0$, is made integrable by the factor $y/(y + f(x))^n$, and determine the form of f(x).

161. By E. B. Seitz. — Two equal circles, radii r, are drawn on the surface of a circle, radius 2r; find the average of the area common to the two circles.

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Query, by W. E. Heal, Wheeling, Indiana. — On page 149 of Chauvenet's Geometry it is stated, "That it is possible, by the use of the straight line and circle only, to construct regular polygons of 17 sides, of 257 sides, and in general of any number of sides which can be expressed by $2^n + 1$, n being an integer, provided that $2^n + 1$ is a prime number." How is this demonstrated?

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ERRATA.

On page 20, line 11, eq. (3), for $(b^2 + a^2)$ in denominator, read $(b^4 + a^4)$.

"22, "7, " $a^2 + b^2$, " " $c^2 + b^2$.

"15, " " $c^2 + a^2$, " " $c^2 + a^2$.

Numerators " $a^2 - b^2$, numerators " $a^2 - b^2$.

39, "7 from bottom, for $nr+n = l_4$, read $nr+m = l_4$.

31, "3 & 4 from bottom, for d read $d$.

52, "1, for $d = d^4 P(4)$, read $d^4 = d^4 P(4)$.

"7, "140800000, in numerator, read 140800000d$^4$.

58, "7, " $-(b^4 + 12d)^4$, " $-4(b^4 + 12d)^4$. 

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DEMONSTRATION OF THE IMPOSSIBILITY OF RESOLVING ALGEBRAICALLY GENERAL EQUATIONS OF A DEGREE SUPERIOR TO THE FOURTH.

TRANSLATED FROM SERRET'S COURS D'ALGEBRE SUPÉRIEURE
BY ALEXANDER EVANS, ESQ., ELKTON, MD.

The properties of the roots of an equation, algebraically resolvable, which we have just demonstrated, have place in all cases whether we are concerned about an equation in which the coefficients have determinate values, or whether we consider the coefficients as indeterminate, and in consequence the roots of the equation as being any quantities whatever, having no dependence among themselves.

Placing ourselves at present in the last point of view, we are about to demonstrate that it is impossible to resolve algebraically, general equations of a degree superior to the fourth.

This theorem was demonstrated for the first time by Abel; but I will present here the very remarkable demonstration of Wantzel.

We will behold in this exact reproduction a merited homage to the memory of a geometer whom death has stricken in the vigor of his talent.

Nevertheless I will suppress certain details, useless here, after the developments which I have given concerning the number of values that a function may acquire.

Let

\[ f(x) = 0 \]

be an equation of the degree \( m \) whose coefficients are indeterminate, and designate by \( x_1, x_2, \ldots, x_m \) its \( m \) roots which we suppose to be algebraically expressible in functions of the coefficients.\(^*\)

"If the equation \( f(x) = 0 \) is satisfied by the value \( x_1 \) of \( x \), whatever its coefficients may be, we ought to reproduce \( x_1 \) identically by substituting in its expression the rational function corresponding to each radical, since the roots of the equation are then entirely arbitrary.\(^*\)

\(^*\)The inverted commas indicate all that is borrowed literally from the memoir of Wantzel.
"In like manner every relation between the roots ought to be identical, and will not cease to exist if we replace in it these roots, the one by the other in any manner whatever.

"Let us designate by \( y \) the first radical which enters into the value of \( x_1 \) in following the order of the calculation, and let \( y^* = p; \) \( p \) will depend immediately upon the coefficients of \( f(x) = 0 \), and will be expressed by a symmetrical function of the roots \( F(x_1, x_2, x_3, \ldots) \); \( y \) will be a rational function \( \varphi(x_1, x_2, x_3, \ldots) \) of the same roots.

"As the function \( \varphi \) is not symmetrical, otherwise the \( n \)th root of \( p \) would be exactly extractable, it ought to change when we permute two roots, \( x_1, x_2 \), for example; but the relation

\[ \varphi^* = F \]

will always be satisfied. Moreover the function \( F \) being invariable in this permutation, the values of \( \varphi \) are among the roots of the equation \( y^* = F \) and we have

\[ \varphi(x_2, x_1, x_3, \ldots) = a\varphi(x_1, x_2, x_3, \ldots) \]

\( a \) being an \( n \)th root of unity.

"If we replace on each side \( x_1 \), by \( x_2 \), and reciprocally, there results

\[ \varphi(x_1, x_2, x_3, \ldots) = a\varphi(x_2, x_1, x_3, \ldots), \]

whence by multiplying in order, \( a^n = 1 \).

"This result proves that the number \( n \), supposed to be prime, is necessarily equal to \( 2; \) therefore the first radical which presents itself in the value of the unknown ought to be of the second degree. This is what happens in effect for the equations which we know how to resolve."

The function \( \varphi \) having only two values changes by any transposition, and will not be changed (see the 19th lesson) by a circular permutation of three or five letters, for these permutations are equivalent to an even number of transpositions.

Let us continue the series of operations indicated in order to form the value \( x_1 \) of \( x \).

"We will combine the first radical with the coefficients of \( f(x) = 0 \), or the function \( \varphi \) with symmetrical functions of the roots, by the aid of the first operations of algebra, and in this way we will obtain a function of the roots susceptible of two values and in consequence invariable by circular permutations of three letters.

"The subsequent radicals will give besides functions of the same kind, if they be of the second degree. Let us suppose that we have reached a radical for which the equivalent rational function may not be invariable by these permutations. Let us always designate it by

\[ y = \varphi(x_1, x_2, x_3, \ldots); \]
in the equation $y^p = p$ we make again

$$p = F(x_1, x_2, x_3, \ldots);$$

this function will be no longer symmetrical, but only invariable by circular permutations of three letters. If we replace $x_1, x_2, x_3, \ldots$ by $x_2, x_3, x_1, \ldots$ in $\varphi$, the relation $\varphi^p = F$ will always subsist; and since $F$ does not change by this substitution, it becomes

$$\varphi(x_2, x_3, x_1, x_4, \ldots) = a \varphi(x_1, x_2, x_3, x_4, \ldots),$$

$a$ designating an $n$th root of unity."

In making in this equation the circular substitution

$$\begin{pmatrix} x_1, x_2, x_3 \\ x_2, x_3, x_1 \end{pmatrix},$$

and repeating this substitution a second time we will have

$$\varphi(x_2, x_1, x_3, x_4, \ldots) = a \varphi(x_3, x_2, x_1, x_4, \ldots);$$

$$\varphi(x_1, x_2, x_3, x_4, \ldots) = a \varphi(x_3, x_2, x_1, x_4, \ldots),$$

and by multiplying the three preceding equations "we conclude that

$$a^n = 1$$

and so $n$ will be equal to 3.

"If the number of the quantities $x_1, x_2, x_3, x_4$, is greater than four, or if the degree of the equation $f(x) = 0$ is higher than the fourth, we will be able to perform in $\varphi$ a circular substitution of five letters in replacing

$$x_1, x_2, x_3, x_4, x_5,$$

by

$$x_2, x_3, x_4, x_5, x_1;$$

the function $F$ will not change, and we will have

$$\varphi(x_2, x_3, x_4, x_5, x_1, \ldots) = a \varphi(x_1, x_2, x_3, x_4, x_5, \ldots);$$

then in repeating the same substitution from all quarters

$$\varphi(x_3, x_4, x_5, x_1, x_2, \ldots) = a \varphi(x_2, x_3, x_4, x_5, x_1, \ldots),$$

By multiplication we obtain

$$x^5 = 1$$

which carries with it

$$a = 1$$

since $a$ is a cube root of unity. So the function $\varphi$ is invariable for the circular permutations of five letters."

Therefore in consonance with a theorem demonstrated in the nineteenth lesson, the function $\varphi$ is also invariable for the circular permutations of three letters.
Thus all the radicals included in the root of a general equation of a
degree superior to the fourth, should be equal to the rational functions of the
invariable roots by the circular permutations of three roots. In substituting
these functions in the expression of $x_1$ we come to an equality of the form
$$x_1 = \psi(x_1, x_2, x_3, x_4, x_5, \ldots),$$
which ought to be identical; but which is impossible since the second mem-
ber remains invariable when we replace $x_1, x_2, x_3$, by $x_2, x_3, x_1$, whilst
the first evidently changes.

Therefore it is impossible to resolve by radicals a general equation of
the fifth degree, or of a higher degree.

The precedent demonstration shows us at the same time that for the
equations of the third and of the fourth degree, the first radical in the order
of the operations ought to be a quadratic radical, and the second a cubic
radical. These circumstances present themselves in effect in the formulas
given by Lagrange and other geometers.”

NOTE.—In order to well understand the developments upon which we are
about to enter, it is necessary to form for ourself a precise idea of the ope-
rations which we have designated by the word substitution.

Let $F(a, b, c, \ldots k, l)$ be a function of $n$ letters. If among these $n$
letters we take $p$ at random, $a, b, c, \ldots g$ for example, and after having
ranged them in a circle, we put each one of them in place of that which
precedes, we say that we have made these $p$ letters suffer a circular permu-
tation, and the substitution

$$\begin{pmatrix} a, b, c, \ldots g \\ b, c, \ldots g, a \end{pmatrix}$$

is called a circular substitution of the order $P$. That being established we
have the following theorem.

Every substitution if it be not circular, is equivalent to several circular sub-
stitutions simultaneously effected upon the different letters.

Let us suppose in effect that we should impose upon the letters $a, b,
c, \ldots f, g$, any substitution whatever; by this substitution $a$ finds itself
replaced by a certain letter $c$ for example, $c$ itself will be replaced by a 3rd
letter $e$, and continuing in this manner we shall fall necessarily upon a let-
ter which will find itself replaced by $a$. But it is evident that the letters
we have thus encountered have undergone a circular permutation.

By taking one of the remaining letters and operating in the same
manner we shall form a new group of letters which will equally have
submitted to a circular permutation, and so on until all the letters are
exhausted.
The reasoning of which we are about to make use gives the means of forming immediately the circular substitutions which are equivalent to a given substitution. Let us consider for example the substitution
\[(a, b, c, d, e, f, g, h, i, j, o)\]
\[(h, o, d, f, b, j, a, g, e, c, i)\]

We shall find that it is equivalent to the three following circular substitutions
\[(a, h, g)\]
\[(b, o, i, e)\]
\[(c, d, f, j)\]

The same proceeding ought also to be employed when we wish to ascertain whether a substitution be circular or not. So we will find that the substitution
\[(a, b, c, d, e, f, g, h, i, j, o)\]
\[(g, d, f, j, a, o, c, e, b, h)\]
is circular, for we may write it in the manner following
\[(a, g, c, f, o, h, i, b, d, j, e)\]
\[(g, c, f, o, h, i, b, d, j, e, a)\]

If after having effected a circular substitution upon \(p\) letters we repeat 1, 2, 3, \ldots \(p\) — 1 times the same substitution, we shall obtain \(p\) different arrangements, but in making this substitution once more we shall reproduce the primitive arrangement.

We designate by the word *transposition* the circular substitution of two letters, that is to say the operation which consists in simply exchanging these two letters the one for the other, and we indicate by the abridged notation \((a, b)\) the transposition of the letters \(a\) and \(b\).

It is evident that every substitution, circular or not, is equivalent to a series of transpositions.

For suppose it concerns us to operate any substitution upon the letters
\[a, b, c, \ldots, f, g\]
we will cause \(a\) to take the new place which it ought to occupy, by a transposition; this being done another transposition will lead \(b\) to the place which it ought to occupy, and so on, until all the letters have taken the places which we wish to assign to them.

The following partial list of references is added for the benefit of students who may wish to consult the original authorities bearing on this subject.


*Serret*: Cours d'algebre superieure, second edition.

*Tchebichef*: Upon the totality of prime numbers between two given limits, and upon Bertrand's postulatum: Note 15 to Serret.
LAGRANGE: Memoirs of the Berlin Academy for 1770; 1771.
VOLSON: Life and works of Cauchy, pages 55—59, 2nd Vol.
BERTRAND: Journal of the polytechnic school, Cahier XXX.
SERRET: Journal of the polytechnic school, Cahier XXXII: Liouville's Journal, Vol. 15: also note 8 to the Cours d'algebre superieure.
HIRSCH, (Meyer): Examples of the literal Calculus and Algebra; translated by Ross: London. 1827.
HIRSCH, (Meyer): Preface to the "Integral Tables" German edition.
JERRARD, . . . "Researches": reduction of the equation of the 5th degree, by means of an equation of the 3rd degree, to the form $x^3 + px + q = 0$: the demonstration is in Note 5, Serret's Algebre Superieure 2nd edition.
NICHOL, . . . : Cyclopedia of the physical sciences; title, Equation, page 348. [As it was thought that the foregoing demonstration, in itself would be found rather obscure and unsatisfactory, we suggested to Mr. Evans that he supply, in the form of a note to the demonstration, for the benefit of such readers as have not access to Serret's book, a translation of so much of the 19th lesson referred to as specially relates to the demonstration under consideration. To that suggestion Mr. Evans has responded with a somewhat extended note.

As our space at present will not permit the introduction of Mr. Evans' note in full, we have inserted above that part of it which relates to circular substitution and transposition, together with the references given by Mr. Evans for the benefit of students who may wish to consult the original discussions bearing upon the subject, and will, in future Nos. of the Analyst, insert any queries that may present themselves to students of this demonstration, and such discussions as the queries may elicit from correspondents who have critically examined the demonstration.—Ed.]
DEMONSTRATION OF A PROPOSITION.

BY THE EDITOR.

Proposition. — The locus of the vertex of a Parabola which rolls on an equal parabola is a Oloid.

The above proposition is announced, it is said, by Prof. Olney in his General Geometry and Calculus, but without demonstration.

It is proposed here to determine the locus of any point in the axis of the rolling parabola, from which the locus of the vertex will be found as a particular case of the more general proposition.

Let $AL$ represent the common vertical tangent to the two parabolas $APQ$ and $AP'Q'$, and let $F$ and $F'$ be their foci, and $F'I$ a directrix through $F'$. Let $P$ be any point in the parabola $APQ$, and $AB$, $BP$ coordinates to the point $P$. Then, putting $AF = a$, $AB = x$ and $PB = y$ we have, from the property of the parabola,

$$y = 2\sqrt{(ax)}$$  \hspace{1cm} (1)

Suppose the parabola $AP'Q'$ to have rolled to the position occupied by the parabola $A'PQ''$, the two parabolas being now tangent at $P$. Draw $PK$ parallel to the axis $AB$ and intersecting the directrix in $F''$; then because, by property of the parabola, $PF'' = PF$, and because $FF''$ is perpendicular to, and is bisected by a tangent at $P$, therefore $F''$ is at the focus of the parabola $A'PQ''$; therefore the focus of the rolling parabola is always on the directrix $F'I$, and consequently the locus of $F'$ is a straight line.

Let $D'$ represent any point in the axis of the parabola $AP'Q'$, at a distance $b$ from the focus $F'$, and let the curve $D'D''$ represent the locus of $D'$; then, $A$ being the origin of coordinates, $AC$ and $CD''$ will be rectangular coordinates to the point $D''$. Put $AC' = CD'' = x'$ and $AC = CD'' = y'$.

Take $FD = F'D' (= b)$ and join $DP$ and $PD''$. Then because in the triangles $DPF$ and $D''PF''$ the angles at $D$ and $D''$ are equal; and because their containing sides $DF$ and $DP$, and $D''F'$ and $D''P$ are equal,
each to each, and their opposite sides $FP$ and $F''P$ are also equal, therefore the triangles are identical; hence the exterior angles $PFB$ and $D''F'H$ are equal. Consequently the triangles $PFB$ and $D''F'H$ are similar. Hence
\[ FB : BP :: F''H : HD'', \text{ or } (a - x) : y :: F''H : HD''. \]
But because $DP = D''P$ and $DF = D'H'$, we have, (Eucl. 12, II),
\[ 2FD \times FB = 2F''P \times F''H, \]
or
\[ 2b(a - x) = 2(a + x)F''H; \]
\[ \therefore F''H = \frac{b(a - x)}{a + x}. \] \hspace{1cm} (2)
Substituting this value for $F''H$ in the above proportion we have
\[ a - x : y :: \frac{b(a - x)}{a + x} : HD''; \]
\[ \therefore HD'' = y \frac{b}{a + x}. \] \hspace{1cm} (3)
Hence we have
\[ x' = F''H + F'A = \frac{b(a - x)}{a + x} + a \] \hspace{1cm} (4)
and
\[ y' = HD'' + BP = y \frac{b}{a + x} + y. \] \hspace{1cm} (5)
From (4) we get
\[ x = \frac{a(b + a - x')}{b - a + x'}, \] \hspace{1cm} (6)
and from (5) and (1),
\[ y' = 2\sqrt{ax}\left(\frac{b}{a + x} + 1\right), \]
or, by substituting for $x$ from (6),
\[ y' = \left(a + b + x'\right)\sqrt{\left(\frac{b + a - x'}{b - a + x'}\right)}, \] \hspace{1cm} (7)
which is the equation to the locus of $D'$. If we take $b = -a$, $D'$ will coincide with $A$, and we shall have for the locus of the vertex
\[ y' = \frac{-x'^5}{x'^2 - 2a} = \sqrt{\frac{x'^5}{2a - x'}}, \]
which is the equation of a cissoid.

Suggestion.—The circumstance that the locus of $F'$ is a straight line suggests various applications of the locus of $D'$ in practical mechanics; as by shifting the point $D'$ we can produce any desired curvature between a straight line and a curve approximating nearly to that of a circle.
SOLUTION OF A PROBLEM.

BY PROF. DAVID TROWBRIDGE, WATERBURGH, NEW YORK.

Problem.—If we put

\[ nQ_a^{(1)} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} \ldots \ldots (1) \]
\[ nQ_a^{(2)} = 1 + Q_a^{(1)} + Q_a^{(1)} + Q_a^{(1)} + \ldots + Q_a^{(1)} \ldots \ldots (2) \]
\[ nQ_a^{(3)} = 1 + Q_a^{(2)} + Q_a^{(2)} + Q_a^{(2)} + \ldots + Q_a^{(2)} \ldots \ldots (3) \]
\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]
\[ nQ_a^{(p)} = 1 + Q_a^{(p-1)} + Q_a^{(p-1)} + \ldots + Q_a^{(p-1)} \ldots \ldots (p) \]
\[ S_a^{(p)} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots + \frac{1}{n^p} \ldots \ldots (4) \]

then

\[ Q_a^{(p-1)} = 1 - \left( n-1 \right) \frac{1}{1.2^p} \frac{(n-1)(n-2)}{1.2.3^p} \frac{(n-1)(n-2)(n-3)}{1.2.3.4^p} \ldots \ldots \ldots \ldots (5) \]

and

\[ S_a^{(p)} = n \left[ n(n-1) \frac{Q_a^{(p-1)}}{1.2} + \frac{n(n-1)(n-2)Q_a^{(p-1)}}{1.2.3} \frac{n(n-1)(n-2)(n-3)Q_a^{(p-1)}}{1.2.3.4} \ldots \ldots \ldots \ldots (6) \right] \]

Solution.—Take the series

\[ 1 + x + x^2 + x^3 + \ldots + x^{n-1} = \frac{x^n - 1}{x - 1} = \frac{(1+y)^n - 1}{y} \]

\[ = n + \frac{n(n-1)}{1.2} y + \frac{n(n-1)(n-2)}{1.2.3} y^2 + \ldots + y^{n-1} \ldots \ldots (7) \]

in which \( z = 1 + y \), which gives \( dx = dy \). Now multiply (7) by \( dx = dy \) and integrate, then

\[ x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots + \frac{1}{n}x^n = C + N_1y + \frac{1}{2}N_2y^2 + \frac{1}{3}N_3y^3 + \ldots + \frac{1}{n}N_ny^n \ldots \ldots (8) \]

\[ 1 + \frac{1}{2}x + \ldots + \frac{1}{n}x^{n-1} = \frac{1}{1+y} \left( C + N_1y + \frac{1}{2}N_2y^2 + \ldots \right) \]

\[ = C(1-y+y^2+) + N_1(y-y^2+y^3) + \frac{1}{2}N_2(y^2-y^3+\ldots+) + \frac{1}{n}N_n(y^n-y^{n+1}+\ldots+) \]

\[ = C(C-N_1)y + (C-N_1+\frac{1}{2}N_2)y^2 - (C-N_1+\frac{1}{2}N_2-\frac{1}{3}N_3)y^3 + \ldots + (-1)^n \frac{1}{n} N_n y^n \]

\[ = A_1 - 2A_2y + 3A_3y^2 - \ldots + (-1)^{n-1} nA_n y^{n-1}. \]
In these equations \( N_1 = n, \quad N_2 = \frac{n(n-1)}{1.2}, \quad N_3 = \frac{n(n-1)(n-2)}{1.2.3}, \quad \&c; \)
\[ A_1 = C, \quad 2A_2 = C - N_1, \quad 3A_3 = C - N_1 + \frac{1}{2}N_2, \quad 4A_4 = C - N_1 + \frac{1}{2}N_2 - \frac{1}{3}N_3, \quad \&c. \ldots (9) \]

By integrating as before we have
\[ x + \frac{1}{2^2}x^2 + \ldots + \frac{1}{n^2}x^n = C + A_1y - A_2y^2 + A_3y^3 - \ldots + (1 - 1)^{n-1}A_n y^n \ldots (10) \]

In a precisely similar manner we shall find
\[ x + \frac{1}{2^2}x^2 + \ldots + \frac{1}{n^2}x^n = C' + B_1y - B_2y^2 + B_3y^3 - \ldots + (1 - 1)^{n-1}B_n y^n \ldots (11) \]
\[ B_1 = C', \quad 2B_2 = C' - A_1, \quad 3B_3 = C' - A_1 - A_2, \quad 4B_4 = C' - A_1 - A_2 - A_3, \quad \&c. \]
\[ x + \frac{1}{2^4}x^2 + \ldots + \frac{1}{n^4}x^n = C'' + E_1y - E_2y^2 + E_3y^3 - \ldots + (1 - 1)^{n-1}E_n y^n \ldots (12) \]
\[ E_1 = C'', \quad 2E_2 = C'' - B_1, \quad 3E_3 = C'' - B_1 - B_2, \quad \&c. \ldots . (13) \]

The integrals (8), (10), (11) and (12) are to be taken between the limits \( x = 1, \quad y = 0, \quad \text{and} \quad x = 0, \quad y = -1. \) These values in (8), (10), (11) and (12) give
\[ S_1^{(1)} = C = N_1 - \frac{1}{2}N_2 + \frac{1}{3}N_3 - \ldots + \frac{1}{n}(1 - 1)^{n-1}N_n \ldots (14) \]
\[ S_1^{(2)} = C' = A_1 + A_2 + A_3 + \ldots + A_n \ldots . . . (15) \]
\[ S_1^{(3)} = C'' = B_1 + B_2 + B_3 + \ldots + B_n \ldots . . . . (16) \]
\[ S_1^{(4)} = C''' = E_1 + E_2 + E_3 + \ldots + E_n \ldots . . . . . . (17) \]

It will now be seen that \( A_{n+1}, \quad B_{n+1}, \quad E_{n+1}, \quad \&c., \) are each equal to 0, so that (8), (10), (11), and (12) are each a finite series. If we eliminate \( A_1, \quad A_2, \quad \&c., \) by means of (9) we shall find
\[ C = C\left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right) - N_1\left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right) - \frac{1}{2}N_2\left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right) \ldots \]
\[ = C - N_1(C - 1) + \frac{1}{2}N_2(C - 1 - \frac{1}{3}) - \frac{1}{3}N_3(C - 1 - \frac{1}{3} - \frac{1}{4}) + \ldots \]
\[ = C - N_1(C - 1) + \frac{1}{2}N_2(C - 2Q_2^{(1)}) - \frac{1}{3}N_3(C - 3Q_3^{(1)}) + \ldots \]
\[ = C - C\left(N_1 - \frac{1}{2}N_2 + \frac{1}{3}N_3 - \ldots + \frac{1}{n}(1 - 1)^{n-1}N_n \right) + N_1 - N_2Q_2^{(1)} + N_3Q_3^{(1)} \ldots \]
\[ = N_1 - N_2Q_2^{(1)} + N_3Q_3^{(1)} - \ldots + (1 - 1)^{n-1}N_n Q_n^{(1)} \ldots . . . . (18) \]

by (14).

In a similar manner we can find
\[ C' = A_1 + 2A_2Q_2^{(1)} + 3A_3Q_3^{(1)} + \ldots + nA_nQ_n^{(1)} \ldots (19) \]
\[ C'' = B_1 + 2B_2Q_2^{(1)} + 3B_3Q_3^{(1)} + \ldots + nB_nQ_n^{(1)} \ldots (20) \]
By eliminating $A_1, A_2, \& c.$ from (19), we shall have
\[ C'' = C(1 + Q_2^{(1)} + \ldots + Q_n^{(1)}) - N_1(Q_2^{(1)} + Q_3^{(1)} + \ldots + Q_n^{(1)}) \]
\[ + \frac{1}{2} N_2(Q_3^{(1)} + Q_4^{(1)} + \ldots + Q_n^{(1)}) - \frac{1}{3} N_3(Q_4^{(1)} + \ldots + Q_n^{(1)}) + \ldots \]
\[ = [\text{by (2)}] nCQ_n^{(2)} - N_1(nQ_n^{(2)} - 1) + \frac{1}{2} N_2(nQ_n^{(2)} - 2Q_n^{(2)}) - \ldots \]
\[ = nQ_n^{(2)}(C - N_1 + \frac{1}{2} N_2 - \ldots + \frac{1}{n}(-1)^n N_n) + N_1 N_2 Q_n^{(2)} + N_3 Q_n^{(3)} - \ldots \]
\[ = N_1 - N_2 Q_2^{(2)} + N_3 Q_3^{(3)} - \ldots + (-1)^{n-1} N_n Q_n^{(n)} \ldots \ldots \ldots \ldots (21) \]

In a similar manner we should find
\[ C''' = N_1 - N_2 Q_2^{(3)} + N_3 Q_3^{(3)} - \ldots + (-1)^{n-1} N_n Q_n^{(n)} \ldots \ldots \ldots (22) \]
and so on for higher values of $p$. If we restore the values of $N_1, N_2, \& c.$, we shall thus have proved the truth of equation (6).

Now to find the value of $Q_n^{(p)}$ we have
\[ nQ_n^{(1)} = N_1 - \frac{1}{2} N_2 + \frac{1}{3} N_3 - \ldots + \frac{1}{n}(-1)^{n-1} N_n \]
\[ = n - \frac{n(n-1)}{1.2^2} + \frac{n(n-1)(n-2)}{1.2.3^2} - \ldots + \frac{1}{n}(-1)^{n-1} \]
\[ \therefore Q_n^{(1)} = 1 - \frac{(n-1)}{1.2^2} + \frac{(n-1)(n-2)}{1.2.3^2} - \ldots + \frac{1}{n^2}(-1)^{n-1} \ldots (23) \]

From this we find
\[ 1 = 1 \]
\[ Q_1^{(1)} = 1 - \frac{1}{1.2^2} \]
\[ Q_2^{(1)} = 1 - \frac{2}{1.2^2} + \frac{1}{1.2.3^2} \]
\[ Q_3^{(1)} = 1 - \frac{3}{1.2^2} + \frac{2}{1.2.3^2} - \frac{1}{1.2.3.4^2} \]
\[ \ldots \ldots \ldots \ldots \]
\[ Q_n^{(1)} = 1 - \frac{n-1}{1.2^2} + \frac{(n-1)(n-2)}{1.2.3^2} - \frac{(n-1)(n-2)(n-3)}{1.2.3.4^2} - \ldots + \frac{1}{n^2}(-1)^{n-1} \ldots (24) \]

If we sum these perpendicularly we shall find
\[ nQ_2^{(2)} = n - \frac{n(n-1)}{1.2^3} + \frac{n(n-1)(n-2)}{1.2.3^3} - \ldots + \frac{1}{n^2}(-1)^{n-1} \ldots (24) \]
\[ \therefore Q_2^{(2)} = 1 - \frac{n-1}{1.2^3} + \frac{(n-1)(n-2)}{1.2.3^3} - \ldots + \frac{1}{n^2}(-1)^{n-1} \ldots (25) \]

It is now plain that we should find in a precisely similar manner
\[ Q_3^{(3)} = 1 - \frac{n-1}{1.2^4} + \frac{(n-1)(n-2)}{1.2.3^4} - \ldots + \frac{1}{n^3}(-1)^{n-1} \ldots (26) \]
and so on for higher values of $p$, $n$ and $p$ being positive integers.
The perpendicular columns above are summed by the formula
\[ \sum [r(r+1)(r+2)\ldots (r+x-1)] = \frac{r(r+1)(r+2)\ldots (r+x)-(r-y)(r-y+1)\ldots (r+1)}{x+1} \]

We have thus given a complete solution of the problem, and found the sum \( S^{(p)} \) in finite terms.

THE ROTATION OF SATURN.

BY ALEXANDER EVANS, ESQ., ELKTON, MARYLAND.

The interesting and valuable article by Professor Asaph Hall has naturally engaged the attention of readers of the Analyst.

That Sir John Herschel should in the ninth edition of his Outlines give the time of rotation of Jupiter as that of Saturn, may be attributed to an accident. But in the English edition of 1849, which may perhaps be called the 2nd, if that published as part 43 of the Cabinet Cyclopædia be called the 1st, the time of rotation is stated to be 10h 29m 17s.

In the Cabinet Cyclopædia edition, at least as republished in America, no time is given.*

Grant in his History of Physical Astronomy, page 252, gives the time as 10h 16m 0.44, derived from one hundred revolutions of the planet: this period was announced by Sir William Herschel in the Philosophical Transactions for 1794, page 62.

It is then extremely singular that Sir John Herschel should in his own first elaborate English edition of 1849 alter the time as announced by his father. Prof. Hall’s conjecture seems very plausible: yet why should Sir John refer to the Systeme du Monde in preference to his father’s original observations?

Prof. Hall having by his computations made such a comparison possible, we may be permitted to see what result would follow from a simple proportion.

Take 1st the Washington observations of December 7th 6h 18m, and that of December 19th at 5h 6m; the interval is one of 286h.8 and estimating by

*I have discovered that the edition of Herschel’s Treatise on astron. by S. C. Walker 1836, taken I believe from No. 24, Cabinet Cyclopædia, does contain the time of rotation of Saturn 10h 29m 17s.

Singularly enough, Humboldt, Cosmos Vol. 4, page 170, in the text, gives 10h 29m 17s, and then says, in a note; “the earliest and careful observations of William Herschel in November 1793, gave for Saturn’s period of rotation 10h 16m 44s.”
Herschel's period, there have been evidently 28 rotations: dividing 286.8 by 28 we have 10° 14′ 24″ for the period of a single rotation, differing two tenths of a second from the determination of Prof. Hall.

Again taking the Washington observations of December 7th 6° 18′, and of January 2nd 7° 9′, both by Prof. Hall, and the first and last in the table, and we have an interval of 624.85 hours with 61 rotations, giving for each rotation a period of 10° 14′ 24″, coinciding with the previous result, and differing from the computation of Professor Hall by two tenths of a second.

[In comparing the period derived from the observed interval between a single pair of observations with that from the adjusted mean of a number of independent observations, the correction, Δt, resulting from the motions of the Earth and Saturn during the observed interval, should have been applied: besides, the quotients in the two cases presented above are respectively, without the correction alluded to, 10h 14m 34s and 10h 14m 36s, instead of 10h 14m 24s* in each case as given above. But even this approximate coincidence must be regarded as accidental, as it is obviously impossible for an observer to determine, by a single observation, the exact time when the center of the spot coincides with the center of the disk of the planet. This fact is recognized in all the observations recorded, as none of the observers have registered their observations in fractions of a minute. Moreover the three contemporaneous observations by Professors Hall, Newcomb and Eastman, who may be assumed to be equally skilful observers, give for the least and greatest difference, respectively, one and five minutes; which shows that the true result can only be obtained with exactness from the adjusted mean of many independent observations.—Ed.]

REMARKS ON PROBLEM 137, BY GEORGE EASTWOOD.—The solution given at page 26 of the Analyst, Vol. IV, may be greatly simplified by assuming the given point D to be at the foot of a perpendicular from A upon the given line BC.

In the annexed diagram, let $AB = a$ given line $= l$, $ABD = a$ given angle $= \omega$, $D$ the given point, and $OD$ the perpendicular from $O$ on $AB = p$, a given line.

Put $\angle OAB = \theta$; then $AD = p \cot \theta$

$BD = p \cot(\omega - \theta)$

$\therefore \quad p \cot \theta + p \cot(\omega - \theta) = l$.

*Mr Evans' result is obtained by limiting the quotient in both cases to two decimals.
\[ \cot \theta + \cot \theta \cot \omega + \frac{1}{\cot \theta - \cot \omega} = \frac{l}{p}, \]

an equation of the second degree in \( \cot \theta \).

\[ \cot \theta = \frac{l}{2p} \pm \frac{1}{2p} \sqrt{\left(4p \cot \omega - 4p^2 + b\right)}. \]

These values of \( \cot \theta \) show that the point \( D \) may be posited at the same distance from \( B \) that it is from \( A \).

The question, under this additional data, is equivalent to the following:

Given the base, the vertical angle, and the perpendicular from the vertical angle upon the base, to construct the triangle.

**Analysis:**—Suppose the thing done, \( AOB \) the required triangle, circumscribed by the circle \( AOFBE \), \( O \) the given vertical angle, \( AB \) the given base, and \( OD \) the given perpendicular. Bisect the base in \( H \), and draw the diameter \( FHE \) parallel to \( OD \). Join \( OF \) meeting \( AB \) in \( G \), and draw \( AE \). Then, since \( AH \) is given, and the angle \( HAE = AOE = \frac{1}{2} \) given vertical angle, \( \therefore AE, HE, \) and \( EF \) are given. Also, since the line \( OE \) bisects the angle \( AOB \), the rectangle under \( AOA \). \( OB = OE. OG = (OG + GE)OG = OG^2 + OG. GE = EF. OD, \) and by a known property \( (OG + GE)GE = GE^2 + OG. GE = AE^2 \). That is \( OG^2 + 2OG. GE + GE^2 = EF. OD + AE^2 \) = sum of two given magnitudes. Make \( EF. OD = a \) a square = \( M^2 \), and \( M^2 + AE^2 = N^2 \); then \( OG^2 + 2OG. GE + GE^2 = N^2 = a \) a given square. Therefore \( OG + GE = \) a given line.

But \( (OG + GE)GE = AE^2 \), therefore \( GE \), and consequently \( OG \), are both given; therefore \( OG. GE = \) a given rectangle.

Again, \( BG.AG = (AB - AG)AG = AB. AG - AG^2 = OG. GE = \) a given magnitude. That is, we are required to take a line from a given line, so that the rectangle under the two parts shall be equal to a given rectangle—a well known geometrical problem. Now \( AG \) being thus determined, \( BG \) will be known. But \( AO : OB :: AG : BG :: 1 : n \), a given ratio. Therefore \( nAO = OB \), and \( nAO^2 = EF. OD = M^2 \), as above. Therefore \( AO \) and \( OB \), the sides of the triangle are known. Whence this

**Construction:** Upon \( AB \) the given base, describe the segment of a circle capable of containing the given angle. At \( D \), the given point, erect the perpendicular \( DO \), meeting the segment in \( O \); join \( AO, BO \), then \( AOB \) is the required triangle. The demonstration is manifest from the analysis.

When the lines \( AO, OB, BA, \) and \( OD \) are arcs of great circles, then, putting angle \( O = \omega \), angle \( AOD = a \), \( OD = p \), and \( AB = \lambda \), we have

\[ \tan AD = \sin p \tan \alpha, \quad \tan BD = \sin p \tan (\omega - a), \]

\[ AD + BD = \tan^{-1} \sin p \tan \alpha + \tan^{-1} \sin p \tan (\omega - a) = \lambda. \quad (1) \]
Therefore, taking tangent function of both sides of (1), we get
\[
\frac{\sin p \tan \alpha + \sin p \tan (\omega - \alpha)}{1 - \sin^2 p \tan \alpha \tan (\omega - \alpha)} = \tan \lambda,
\]
or, by dropping functional characteristics of \( \alpha, \omega \) and \( \lambda \),
\[
\sin p \left( \alpha + \frac{\omega - \alpha}{1 + \omega \alpha} \right) + \left( 1 - \sin^2 p \frac{\omega \alpha - \alpha^2}{1 + \omega \alpha} \right) = \lambda. \quad \cdots \quad (2)
\]
By expansion and reduction, (2) becomes
\[
(\omega \sin p - \lambda \sin^2 p) \alpha^2 - \lambda \omega \cos^2 p \alpha = \lambda - \omega \sin p. \quad \cdots \quad (3)
\]
The two roots of \( \alpha \) in (3) are
\[
\frac{\omega \lambda \cos^2 p}{2(\omega \sin p - \lambda \sin^2 p)} + \frac{[4(\lambda - \omega \sin p)(\omega \sin p - \lambda \sin^2 p) + \lambda \omega \cos^2 p]}{2(\omega \sin p - \lambda \sin^2 p)}, \quad \text{and}
\]
\[
\frac{\omega \lambda \cos^2 p}{2(\omega \sin p - \lambda \sin^2 p)} - \frac{[4(\lambda - \omega \sin p)(\omega \sin p - \lambda \sin^2 p) + \lambda \omega \cos^2 p]}{2(\omega \sin p - \lambda \sin^2 p)},
\]
which shows, as in a plane, that the point \( D \) may be posited at the same distance from \( B \) that it is from \( A \).

This problem, in its most general form, has a somewhat remarkable history. A general solution may be found in Newton's Universal Arithmetic, prob. 24; and in Gergonne, Annales de Mathematiques, tom. 10, p. 205.

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ON ADJUSTMENT FORMULAS.

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BY E. L. DE FOREST.

WHEN a series of observed numbers shows irregularities resulting from errors of observation, the simplest mode of correcting them is by graphical construction, plotting the terms of the series on paper as ordinates to a curve, drawing a smooth curve through the points thus obtained so as to coincide with them as nearly as may seem best, and then measuring the ordinates thus corrected. This will often give results accurate enough for practical purposes. It also has the advantage of bringing the conditions of the problem together before the eye, so that graphical construction will always be a valuable adjunct and preliminary to the use of other methods which may be expected to give a higher degree of accuracy.

Of the analytical methods for adjusting a series, the simplest and most easily applied by persons of moderate mathematical knowledge is that which presupposes that the terms of the given series are equidistant, or if not, that they have been reduced to equidistant ones either graphically or by simple interpolation, and then adjusts the middle term \( u_o \) of any group of
$2m+1$ terms by means of the formula

$$w'_0 = l_0 u_0 + l_1 (u_1 + u_{-1}) + l_2 (u_2 + u_{-2}) + \ldots + l_m (u_m + u_{-m}),$$ (1)

where $l_0, l_1, l_2 \text{ etc.}$ are numerical coefficients determined in advance. It is most advantageous to assume that the true law of the series is algebraic and of a degree not higher than the third, so that if there were no errors of observation, the fourth and higher orders of differences would be zero. In point of fact, the true analytical law is usually unknown, but the above assumption respecting it will be practically safe, provided that the form of the given series, when plotted on paper is seen to approximate to that of an algebraic curve of the third degree, or a lower degree, throughout every group of $2m+1$ consecutive terms. Such a curve can have only one point of inflexion, so that the method is obviously unsuited to the adjustment of series which are known to be of sinuous form, so as to have very frequent changes of flexure.

Subject to the condition named, there may be an infinite number of systems of values assigned to the coefficients $l$, such that if the adjustment formula is applied to a series of the third or any lower order, the adjusted value $w'_0$ will be the same as the given value $u_0$. Schiaparelli has investigated some properties of these formulas, with a view to the reduction of meteorological observations chiefly, in his treatise *Sul modo di ricavare la vera espressione delle leggi della natura dalle curve empiriche*; Milan, 1867. He showed that the coefficients $l$ must be connected by these two relations;

$$\begin{align*}
l_0 + 2(l_1 + l_2 + \ldots + l_m) &= 1, \\
1^2 l_1 + 2^2 l_2 + 3^2 l_3 + \ldots + m^2 l_m &= 0. \end{align*}$$ (2)

He also ascertained what the numerical values of $l$ must be, for values of $2m+1$ from 5 to 11 inclusive, in order that the probable errors of the adjusted terms may be reduced to a minimum. The results thus reached are really identical with those obtained by the method of least squares, when the $2m+1$ terms are each supposed to satisfy approximately the equation

$$u = a + bx + cx^2 + dx^3$$

and are also regarded as of equal weight, that is, equally liable to error. (Compare an article by the present writer, in the *Smithsonian Report* of 1871; pp. 326 and 336.) It can be shown that the adjustment formulas thus constructed are embraced under the general form

$$\begin{align*}
[(2m+1)(m^4) - 2(m^4)(m^2)] u'_0 &= (m^4) u_0 + [(m^4) - 1^2(m^4)] (u_1 + u_{-1}) \\
&+ [(m^4) - 2^2(m^4)] (u_2 + u_{-2}) + \ldots \ldots \ldots \\
&+ [(m^4) - m^2(m^2)] (u_m + u_{-m}) \end{align*}$$ (3)

where the notation used is

$$(m^r) = 1^r + 2^r + \ldots + m^r.$$
The values of the coefficients $l$ for all values of $2m+1$ from 5 to 25 inclusive, are here given in Table A. They are only carried out to two or three places of decimals, which are all that are needed in practice, and the last decimal figure is in some cases altered by a single unit in order that the condition $l_0 + 2(l_1 + l_2 + \ldots + l_n) = 1$ may be exactly satisfied.

**Table A.**

*(giving minimum values to $\epsilon'/\epsilon$.)*

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<th>$2m+1$</th>
<th>5</th>
<th>7</th>
<th>9</th>
<th>11</th>
<th>13</th>
<th>15</th>
<th>17</th>
<th>19</th>
<th>21</th>
<th>23</th>
<th>25</th>
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<td>.26</td>
<td>.208</td>
<td>.174</td>
<td>.152</td>
<td>.134</td>
<td>.118</td>
<td>.108</td>
<td>.098</td>
<td>.090</td>
</tr>
<tr>
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<td>.34</td>
<td>.29</td>
<td>.23</td>
<td>.196</td>
<td>.168</td>
<td>.147</td>
<td>.130</td>
<td>.117</td>
<td>.106</td>
<td>.097</td>
<td>.089</td>
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<tr>
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<td>.17</td>
<td>.161</td>
<td>.147</td>
<td>.133</td>
<td>.121</td>
<td>.110</td>
<td>.101</td>
<td>.093</td>
<td>.086</td>
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<td>.102</td>
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<tr>
<td>$l_7$</td>
<td>-.071</td>
<td>-.019</td>
<td>.011</td>
<td>.027</td>
<td>.037</td>
<td>.043</td>
<td></td>
<td></td>
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<td></td>
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<tr>
<td>$l_8$</td>
<td>-.065</td>
<td>-.023</td>
<td>.003</td>
<td>.019</td>
<td>.028</td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>$l_9$</td>
<td>-.060</td>
<td>-.025</td>
<td>.002</td>
<td>.012</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>$l_{10}$</td>
<td>-.056</td>
<td>-.028</td>
<td>.006</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$l_{11}$</td>
<td>-.052</td>
<td>-.027</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$l_{12}$</td>
<td>-.049</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\epsilon'/\epsilon$</td>
<td>.69</td>
<td>.58</td>
<td>.50</td>
<td>.456</td>
<td>.418</td>
<td>.389</td>
<td>.365</td>
<td>.345</td>
<td>.328</td>
<td>.313</td>
<td>.300</td>
</tr>
<tr>
<td>$(\Delta_4)$</td>
<td>.29</td>
<td>.19</td>
<td>.13</td>
<td>.109</td>
<td>.0907</td>
<td>.0782</td>
<td>.0677</td>
<td>.0600</td>
<td>.0548</td>
<td>.0496</td>
<td>.0456</td>
</tr>
</tbody>
</table>

The probable error of a term adjusted by any such formula as (1), assuming that the true law of the series is algebraic and of a degree not higher than the third, will be

$$\epsilon' = \sqrt{[l_0^2 \epsilon_0^2 + l_1^2 (\epsilon_1^2 + \epsilon_2^2) + \ldots + l_n^2 (\epsilon_n^2 + \epsilon_{n-1}^2)]}$$

where $\epsilon_0$, $\epsilon_1$ &c., denote the probable errors of the given terms $u_0$, $u_1$ &c. If these are all equal, denoting their common value by $\epsilon$ we shall have

$$\frac{\epsilon'}{\epsilon} = \sqrt{[l_0^2 + 2(l_1^2 + l_2^2 + \ldots + l_n^2)]}, \ldots \ldots \ldots (4)$$

and this may be regarded as approximately the ratio which the probable error of any adjusted term will bear to that of the corresponding unadjusted one, so far as this ratio can be determined a priori. It is true that the errors of the given terms will not in general be really equal, but it usually
happens in statistical and other series that the weights of the terms follow a tolerably regular sequence, so that if for instance the errors \( \varepsilon_1, \varepsilon_2, \ldots \), are greater than \( \varepsilon_0 \), the errors \( \varepsilon_{-1} \) and \( \varepsilon_{-2} \) will be less than \( \varepsilon_0 \), and vice versa, and thus they tend to correct each other. The values of \( \varepsilon^2 \) too are much larger for the middle of a formula than for its extreme terms, so that the error \( \varepsilon' \) is chiefly due to the errors of \( \varepsilon_0 \) and a few other terms adjacent to it. The values of \( \varepsilon' + \varepsilon \) corresponding to the several values of \( 2m + 1 \) in Table A. are given at the foot of that table. These are minimum values, for the reason that, under the conditions we have assumed, the method of least squares gives the most probable value of the adjusted term \( \varepsilon' \).

Before having met with Schiaparelli's work the present writer had constructed, with special reference to the adjustment of mortality tables, various systems of formulas of this general character, one of the most advantageous of which seemed to be that which renders the probable values of the fourth differences of the adjusted series a minimum. (Smithsonian Report of 1871, p. 332.) The series is thus graduated with the greatest possible smoothness. The coefficients under this system are here given in Table B. In a continuation of the article just cited, (Sm. Rep. of 1873, p. 327), the subject was still farther investigated. The fourth difference of any five consecutive terms in the given series being

**Table B.**

\[
\begin{array}{|c|ccccccccccc|}
\hline
2m+1 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 19 & 21 & 23 & 25 \\
\hline
l_0 & .56 & .42 & .34 & .290 & .252 & .222 & .200 & .182 & .166 & .1534 & .1424 \\
l_1 & .29 & .29 & .27 & .245 & .222 & .202 & .184 & .170 & .157 & .1463 & .1368 \\
l_2 & -.07 & .05 & .11 & .135 & .145 & .146 & .143 & .137 & .132 & .1282 & .1205 \\
l_3 & -.05 & -.02 & .026 & .056 & .076 & .087 & .093 & .096 & .0969 & .0962 \\
l_4 & -.03 & -.028 & -.007 & .015 & .034 & .047 & .057 & .0635 & .0678 \\
l_5 & -.03 & -.027 & -.018 & -.004 & .010 & .022 & .0317 & .0395 \\
l_6 & -.015 & -.022 & -.020 & -.012 & -.003 & .0065 & .0151 \\
l_7 & -.010 & -.017 & -.018 & -.015 & -.0092 & -.0024 \\
l_8 & -.007 & -.013 & -.016 & -.015 & -.0119 \\
l_9 & -.005 & -.010 & -.013 & -.0140 \\
l_{10} & -.003 & -.0077 & -.0110 \\
l_{11} & -.0025 & -.0060 \\
l_{12} & -.0018 & \\
\hline
\varepsilon' & .70 & .60 & .54 & .495 & .461 & .433 & .409 & .389 & .372 & .358 & .345 \\
\varepsilon & \\
\hline
(\Delta') & .23 & .080 & .038 & .0187 & .0104 & .0064 & .00422 & .00273 & .00185 & .00124 & .00091 \\
\hline
\end{array}
\]
\[ D_4 = 6u_0 - 4(u_1 + u_{-1}) + (u_2 + u_{-2}) \]

its probable value, without regard to sign, is

\[ (D_4) = \varepsilon_1 / [6^2 + 2(4^2 + 1)] = \varepsilon_1 / 70, \]

and hence we have

\[ \varepsilon = .11952(D_4). \]

It was shown (p. 330) that the probable value of the fourth differences of the adjusted series is

\[ (D'_4) = \varepsilon_1 / [D_5^2 + 2(D_4^2 + D_3^2 + \ldots + D_{-2}^2)], \]

where \( D_0, D_1, D_2 \text{ &c.} \) are the fourth differences of the series

\[ l_2, l_1, l_0, l_1, l_2, \ldots , l_m, 0, 0, 0, 0. \]

Combining (5) and (6), we get

\[ \frac{(D'_4)}{(D_4)} = .11952\sqrt{[D_5^2 + 2(D_4^2 + D_3^2 + \ldots + D_{-2}^2)]}, \]

which enables us to compute, for any adjustment formula such as (1), the ratio which the probable value of any fourth difference of the adjusted series bears to that of the corresponding fourth difference in the unadjusted series. The values of this ratio for Tables A. and B. are found at the bottom of the tables. They measure the smoothness of graduation. For instance, the ratio .00091 for the 25-term formula of Table B. shows that when that formula is used the adjusted series will be smoothly graduated to three places of decimals farther than the given series is.

An error was committed in connection with this subject, which had escaped my notice until quite recently, and I take this earliest opportunity to correct it. Since we have for the given series the relation

\[ \varepsilon = .11952(D_4) \]

it was too hastily inferred that we shall also have for the adjusted series

\[ \varepsilon' = .11952(D'_4) \]

and consequently

\[ \frac{\varepsilon'}{\varepsilon} = \frac{(D'_4)}{(D_4)} \]

from which the conclusion was drawn that the coefficients \( l \) in Table B. (called Table III. in the article referred to) must render the ratio \( \varepsilon' + \varepsilon \) a minimum, in as much as they are known to render \( (D'_4) + (D_4) \) a minimum. It was thus made to appear that the formulas of Table B. are the ones which give the most probable value to the adjusted term \( u'_{-2} \), for any given value of \( 2m + 1 \), and that the ratio \( (D'_4) / (D_4) \) is, in theory at least, equal to the ratio of the probable error of the adjusted term to that of the given one. A great discrepancy between this result and the facts of the case was
indeed recognized at the time and remarked upon in the context, but it was accounted for by the existence of causes which were really of minor importance, the true cause, an error of theory, being overlooked. The truth is that, according to the theory of probabilities, the process by which we obtained the relation

$$
\varepsilon = .11952(\Delta_4)
$$

implies that the errors of the terms $u_0, u_1, \&c.$, of the given series are wholly independent of each other, and that errors in excess and errors in defect are equally likely to occur. This is really true for the terms of the given series, but it is not true for the adjusted series, because any five consecutive adjusted terms have been all computed, to a considerable extent, from the same given terms, only taken in somewhat different proportions, so that the errors of these adjusted terms will most probably be all in the same direction, that is, all in excess or all in defect. Hence the relation

$$
\varepsilon' = .11952(\Delta'_4)
$$

does not hold good, even approximately, and any deductions from it must fall to the ground. The formulas given in pages 330 and 331 of the *Smithsonian Report* of 1873 will be corrected if we substitute $(\Delta'_4)\div(\Delta_4)$ for $\varepsilon' = \varepsilon$ where ever the latter occurs. So also the value 0.232 on page 332 really represents the ratio $(\Delta'_6)\div(\Delta_6)$, and the values 0.305 and 0.460 on page 333 represent the ratio $(\Delta'_{2+2})\div(\Delta_{2+2})$. They measure the smoothness of the adjustment, but they do not measure its accuracy. And it is not true as stated at the foot of p. 337, that when repeated adjustments are made, the ratio of the probable error of the final series to that of the original one will be the product of the ratios due to the two or more formulas used. To find the probable error in such a case, we shall have to express the final adjusted term as a linear function of the original terms, and then proceed as in the case of formulas (1) and (4).

The correct values of the ratio of error $\varepsilon' = \varepsilon$, as found by formula (4) from the coefficients in Table B., are given at the foot of that Table. They are somewhat larger than those of Table A., while the ratios of irregularity $(\Delta'_4)\div(\Delta_4)$ are a good deal smaller. It is impossible to get at once the maximum of probable accuracy and the maximum of smoothness. Formulas may, however, be constructed so as to give intermediate and moderately small values to both ratios. Very good results are obtained by assuming that the coefficients $l$ are ordinates to the curve

$$
y = A + Cx + Dx^2 + Ex^3,
$$

the constant interval between these ordinates being taken as the unit of $x$. The values of the four constants are found from the four equations.
\[
\begin{align*}
(2m+1)A + 2[(m^3)C+(m^3)D+(m^3)E] &= 1 \\
(m^3)A+(m^3)C+(m^3)D+(m^3)E &= 0 \\
A &+ (m+1)^3 C + (m+1)^3 D + (m+1)^3 E = 0 \\
2C + 3(m+1)^3 D + 4(m+1)^3 E &= 0
\end{align*}
\]

(8)

The first two equations follow from Schiaparelli’s conditions (2) already given, while the last two require that \( y \) and \( \frac{dy}{dx} \) should both become zero when \( x = \pm (m + 1) \). (Compare the Sm. Reports of 1871 and 1873, pp. 322 and 352.) Having obtained the numerical values of \( A, C, D \) and \( E \) for any assumed value of \( 2m + 1 \), and substituted them in the equation of the curve, we assign to \( x \) the values 0, 1, 2, \ldots, \( m \), in succession, and the resulting values of \( y \) will be \( l_0, l_1, l_2, \ldots, l_m \). These are here given in Table C., for values of \( 2m+1 \) from 5 to 25. The ratios \( \epsilon' = \epsilon \) are but little larger than those in Table A., while the ratios \( (A'_4) = (A_4) \) are considerably smaller, excepting for the very shortest formulas, which are of no great consequence.

**Table C.**

<table>
<thead>
<tr>
<th>(2m+1)</th>
<th>5</th>
<th>7</th>
<th>9</th>
<th>11</th>
<th>13</th>
<th>15</th>
<th>17</th>
<th>19</th>
<th>21</th>
<th>23</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>(l_0)</td>
<td>.44</td>
<td>.32</td>
<td>.24</td>
<td>.206</td>
<td>.176</td>
<td>.154</td>
<td>.136</td>
<td>.122</td>
<td>.112</td>
<td>.102</td>
<td>.094</td>
</tr>
<tr>
<td>(l_1)</td>
<td>.37</td>
<td>.30</td>
<td>.24</td>
<td>.204</td>
<td>.176</td>
<td>.154</td>
<td>.136</td>
<td>.122</td>
<td>.112</td>
<td>.102</td>
<td>.094</td>
</tr>
<tr>
<td>(l_2)</td>
<td>-.09</td>
<td>-.14</td>
<td>.18</td>
<td>.177</td>
<td>.162</td>
<td>.146</td>
<td>.133</td>
<td>.121</td>
<td>.110</td>
<td>.102</td>
<td>.094</td>
</tr>
<tr>
<td>(l_3)</td>
<td>-.10</td>
<td>-.03</td>
<td>.093</td>
<td>.117</td>
<td>.121</td>
<td>.118</td>
<td>.111</td>
<td>.104</td>
<td>.098</td>
<td>.091</td>
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</tr>
<tr>
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<td>.038</td>
<td>.070</td>
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<td>.090</td>
<td>.090</td>
<td>.088</td>
<td>.085</td>
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<td>.073</td>
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<td>-.012</td>
<td>-.015</td>
<td>-.034</td>
<td>-.047</td>
<td>-.054</td>
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</tr>
<tr>
<td>(l_7)</td>
<td>-.033</td>
<td>-.039</td>
<td>-.021</td>
<td>-.001</td>
<td>.018</td>
<td>.031</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>(l_8)</td>
<td>-.026</td>
<td>-.035</td>
<td>-.025</td>
<td>-.009</td>
<td>-.007</td>
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<td></td>
</tr>
<tr>
<td>(l_9)</td>
<td>-.020</td>
<td>-.032</td>
<td>-.027</td>
<td>-.015</td>
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<td></td>
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<tr>
<td>(l_{10})</td>
<td>-.016</td>
<td>-.028</td>
<td>-.026</td>
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</tr>
<tr>
<td>(l_{11})</td>
<td>-.013</td>
<td>-.024</td>
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<tr>
<td>(l_{12})</td>
<td>-.011</td>
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<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\epsilon' &= .70, .58, .50, .461, .427, .398, .375, .354, .339, .325, .311 \\
\frac{A'_4}{A_4} &= .36, .19, .088, .0516, .0324, .0207, .0146, .0103, .00802, .00589, .00480
\end{align*}
\]

The question as to which of the formulas of the three tables A, B or C is most advantageous for use in any given case, would be easily answered if the conditions we have assumed were really fulfilled in practice, that is to say, if the series of statistical or physical observations which we wish to
adjacent were really of algebraic form and of a degree not higher than the third, subject only to accidental deviations from this law. The best formula would then be the 25-term formula of Table A., since this has the smallest ratio of error \( e' + e \). But actually, the true law of an observed series will probably be quite unknown, and we can only assume that a group of \( 2m + 1 \) terms will follow the algebraic law approximately, and this assumption may not be safe for so many as 25 consecutive terms. The effect of a change in the form of the function, or law of the series, may be illustrated thus:

Take the following equidistant values of the function

\[ u = \sin \theta \]

which is not of algebraic form.

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( u )</th>
<th>( \theta )</th>
<th>( u )</th>
<th>( \theta )</th>
<th>( u )</th>
</tr>
</thead>
<tbody>
<tr>
<td>15°</td>
<td>.25882</td>
<td>35°</td>
<td>.57358</td>
<td>60°</td>
<td>.86603</td>
</tr>
<tr>
<td>20°</td>
<td>.34202</td>
<td>40°</td>
<td>.64279</td>
<td>65°</td>
<td>.90631</td>
</tr>
<tr>
<td>25°</td>
<td>.42282</td>
<td>45°</td>
<td>.70711</td>
<td>70°</td>
<td>.93969</td>
</tr>
<tr>
<td>30°</td>
<td>.50000</td>
<td>50°</td>
<td>.76604</td>
<td>75°</td>
<td>.96593</td>
</tr>
</tbody>
</table>

Apply to these values the 11-term formula of Table A. and the 13-term formula of Table B. The two adjusted values of the middle term \( \sin 45° \) will be .70702 and .70712 which differ from the true value by .00009 and .00001.

The formula from Table B. gives the best result, although its error-ratio \( e' + e \) is a little the largest. The reason is mainly that \( t_0 \) and \( t_1 \) are larger, and \( t_m \) is smaller, in that formula than in the other, giving a greater weight to the middle term and terms adjacent to it. Thus the formulas of Table B. follow the actual curvature of the given series more closely than those of Table A. It seems advisable not to be governed solely by the theoretical ratios \( e' + e \), but to employ such formulas as appear advantageous on various grounds, and rest content if the series thus adjusted is a smoothly graduated one, and satisfies certain tests of good adjustment, that is, certain conditions which would probably be satisfied by the true series, if we had it. There is some advantage in having a series graduated smoothly, even if the probable errors are such that two or three of the last decimal figures are in doubt, and cannot be regarded as having strictly any real value. The chief advantage is, that errors of computation can be more readily detected by the irregularity they produce in the differences of the series.

*(To be concluded in No. 4.)*
RECENT MATHEMATICAL PUBLICATIONS.

COMMUNICATED BY G. W. HILL.


NOTE ON A LOGARITHMIC SERIES.

BY W. E. HEAL, WHEELING, INDIANA.

The common logarithmic series, given in all standard works on Algebra, may, by a simple substitution, be transformed into another that converges more rapidly. I do not remember of seeing the series given in any work that has come under my notice, though it may be well known by mathematicians. I think it would be well to have this series introduced into our Algebras and taught in our schools.

Take the series

\[ \log (z + 1) = \log z + 2M \left[ \frac{1}{2z+1} + \frac{1}{3(2z+1)} + \frac{1}{5(2z+1)^3} \cdots \right]. \]

Let \( z = x^2 - 1 \) and we have

\[ \log x^2 = \log (x^2 - 1) + 2M \left[ \frac{1}{2x^2-1} + \frac{1}{3(2x^2-1)} + \frac{1}{5(2x^2-1)^3} \cdots \right]. \]

But \( \log (x^2-1) = \log (x+1) + \log (x-1) \) and \( \log x^2 = 2 \log x \). Therefore

\[ \log(x+1) = 2 \log x - \log(x-1) - 2M \left[ \frac{1}{2x^2-1} + \frac{1}{3(2x^2-1)} + \frac{1}{5(2x^2-1)^3} \cdots \right]. \]
SOLUTION OF PROBLEMS IN NUMBER TWO.

Solutions of problems in number 2 have been received as follows:

From S. L. Curry, 151; M. B. W. Granger, 151; Henry Heaton, 160 and 161; Prof. E. W. Hyde, 156 and 160; Prof. A. Hall, 160; G. B. Halsted, 156; Chas. H. Kummell, 155 and answer to Prof. Hall's query; Prof. W. W. Johnson, 156 and 158; Christine Ladd, 157; Artemas Martin, 159 and 161; Dr. A. B. Nelson, 151 and 156; W. L. Marcy, 154, 158 and 159; J. B. Mott, 153; O. H. Merrill, 151; Prof. Orson Pratt, 154; Prof. J. Scheffer, 151, 153, 154, 158 and 159; E. B. Seitz, 159 and 161; C. T. Thompson, 151. Dr. A. B. Nelson, Alex. Evans, and T. P. Stowell, each answered Mr. Heal's query, and R. J. Adcock and W. Stille, Prof. Hall's.

151. "What is the altitude of the maximum cylinder which can be inscribed in a given paraboloid?"

SOLUTION BY O. H. MERRILL, SOUTH RUTLAND, N. Y.

Let $a$ be the altitude of the given parabola, $p$, its parameter, $x$ the distance from its vertex to the upper extremity of the required cylinder and $r$ the radius of the cylinder. Then we have

$$r^2 = px; \therefore \pi px(a - x) = \text{vol. of cyl.} = \text{a max.}; \therefore ax - x^2 = \text{a max.}$$

Putting the first differential coefficient equal to zero we get $x = \frac{a}{2}.$

152. "Find the surface of a right conoid with a circular base."

[No solution of this question has been received. — If we represent the required surface by $S$, we shall have

$$S = \int \int dx \, dy \left[ 1 + \left( \frac{dz}{dx} \right)^2 + \left( \frac{dz}{dy} \right)^2 \right] \frac{1}{2}.$$ For the equation of this surface see p. 126, Vol. II of this Journal.—Ed.]

153. "Prove that if $(1 + 0 \times \frac{1}{3} \sqrt{y} - 1)^{1+0} = (1 - 0 \times \frac{1}{3} \sqrt{y} - 1)^{1+0} = \sqrt{1 - 1} \ldots (1), x = [(2 \sqrt{3} - 1 - 0) [1 + \sqrt{1 - 1}^0] - (1 - \sqrt{1 - 1}^0)]; \ldots (2) and find the real approximate value of $x.$"

SOLUTION BY PROF. J. SCHEFFER, COLLEGE OF ST. JAMES, MD.

Developing $(1 + y \times \frac{1}{3} \sqrt{y} - 1)^{1\sqrt{y}} - (1 - y \times \frac{1}{3} \sqrt{y} - 1)^{1\sqrt{y}}$ by the Binomial Theorem and putting $y = 0$ in the resulting series, we get $2\sqrt{1 - 1} \times \sin \frac{1}{3}.$

Hence we have, by (2), $2\sqrt{1 - 1} \times \sin \frac{1}{3} = \sqrt{1 - 1}; \therefore \sin \frac{1}{3} = \frac{1}{2}; \therefore x = \pi.$
Developing \((1 + \sqrt{-1})^y - (1 - \sqrt{-1})^y\) and putting \(y = 0\) in the resulting series we get \([((1 + \sqrt{-1})^y - (1 - \sqrt{-1})^y) + 0 = 2\sqrt{-1} \times 1\pi\). Consequently

\[ x = \pi = [(2\sqrt{-1} \div 0)][(1 + \sqrt{-1})^y - (1 - \sqrt{-1})^y].\]

154. "Find a general logarithmic theorem for the differentiation of

\[ u = z^1 x^2 ...^n \]

\(z, x_1, x_2, \& c\), being any functions of one variable as \(x\)."

**SOLUTION BY PROF. SCHEFFER.**

Putting \(u = z^1; v_1 = x_1^1; v_2 = x_2^2; v_3 = x_3^3, \& c\), we find, by taking natural logarithms, \(\log u = v_1 \log z; \log v_1 = v_2 \log x_1; \log v_2 = v_3 \log x_2, \& c\). Differentiating each of these equations:

\[
\frac{du}{u} = \frac{dv_1}{v_1} = \frac{dx_1}{x_1} \log z; \quad \frac{dv_2}{v_2} = \frac{dx_2}{x_2} \log x_1; \quad \frac{dv_3}{v_3} = \frac{dx_3}{x_3} \log x_2, \& c.
\]

etc. Substituting the values of \(dv_1, dv_2, \& c\), we obtain

\[
\frac{du}{u} = \frac{dv_1}{v_1} = \frac{dv_2}{v_2} = \frac{dv_3}{v_3} = \frac{dx_1}{x_1} + v_1 v_2 \log z \frac{dx_2}{x_2} + v_1 v_2 v_3 \log z \log x_1 \frac{dx_3}{x_3} + ... \]

from which the law is apparent. We may put this expression in the form

\[
\frac{du}{u} = v_1 \left\{ \frac{dx_1}{x_1} + v_3 \log z \left( \frac{dx_2}{x_2} + v_4 \log z \log x_1 \frac{dx_3}{x_3} + ... \right) \right\}.
\]

[The foregoing solution is substantially the same as the solutions sent by Prof. Pratt and Mr. Marcy, but we have adopted Prof. Scheffer's notation on account of its being the most convenient for the printer.]

155. [A solution of this question by Chas. H. Kummell will be published in No. 4.]

156. "In a determinant of the \(n\)th degree the elements of the principal diagonal consist of units, and of the remaining elements those in the first column are each equal to \(a\), those in the second column each equal to \(b\) and so on. Evaluate the determinant."

**SOLUTION BY PROF. JOHNSON.**

The determinant is a symmetrical function of the letters since any two of them may be interchanged by interchanging two rows and at the same time the corresponding columns. Its principal term is unity, and every other term contains at least two letters. Hence denoting the determinant by \(D\) we may at once assume

\[ D = 1 + A \Sigma ab + B \Sigma abc + \ldots.\]
Now if each of two of the letters equals unity, say if \( k = l = 1 \) we shall have \( A = 0 \), since two columns will thus be rendered identical. Denoting by \( \Sigma' \) the sums taken for \( n - 2 \) letters, \( k \) and \( l \) being excluded; we have when \( k = l = 1 \), \( \Sigma ab = \Sigma'ab + 2\Sigma'a + 1 \), \( \Sigma abc = \Sigma'abc + 2\Sigma'ab + \Sigma'a \), &c. Hence putting \( k = l = 1 \), the assumed equation becomes

\[
0 = 1 + A + (2A + B)\Sigma'a + (A + 2B + C)\Sigma'ab + (B + 2C + D)\Sigma'abc + \ldots,
\]

which must hold independently of the values of \( a, b, c \) &c. Therefore equating to zero the coefficients we derive \( A = 1 \), \( B = 2 \), \( C = -3 \), &c., and we have

\[
A = 1 - \Sigma ab + 2\Sigma abc - 3\Sigma abcd + \ldots.
\]

157. "What is the entire number of double points which can be assumed arbitrarily on a curve of the \( n \)th degree?"

**SOLUTION BY CHRISTINE LADD, UNION SPRINGS, N. Y.**

A curve of the \( n \)th degree can be subjected to \( \frac{1}{2}n(n + 3) \) conditions. To require that a fixed point be a double point is equivalent to three conditions; hence the number of double points which can be assumed arbitrarily is never greater than \( \frac{1}{2}n(n + 3) \). In some cases it is not so great.

The number of double points possible to a proper \( n \)-ic is \( \frac{1}{2}(n - 1)(n - 2) \). When this number is less than \( \frac{1}{2}n(n + 3) \), that is, when \( n < 6 \), the curve cannot have so many as \( \frac{1}{2}n(n + 3) \) double points, but as many as it can have at all can be taken arbitrarily. When \( n > 5 \), it is necessary to consider whether the assumption of \( \frac{1}{2}n(n + 3) \) double points will not cause the \( n \)-ic to break up into two curves of lower degree. For instance, if \( n = 6 \), \( \frac{1}{2}n(n + 3) = 9 \), but through these nine points can be passed a determinate cubic, and, in general the only sextic having these points for double points is the cubic twice repeated. (Salmon, Higher Plane Curves, Art. 45.) But, in fact, 6 is the only value of \( n \) for which, in general, the curve can break up into two curves of lower degree, each passing through the \( \frac{1}{2}n(n + 3) \) double points. That number of double points will consume all the conditions or all but two of them according as \( n \) is or is not a multiple of three. When \( n \) is odd, the \( n \)-ic may break up into a \( \frac{1}{2}(n + 1) \)-ic and a \( \frac{1}{2}(n - 1) \)-ic. But through a number of points by which a \( \frac{1}{2}(n - 1) \)-ic can be determined, or through two more than that number, a \( \frac{1}{2}(n + 1) \)-ic cannot pass, and by the number of points through which a \( \frac{1}{2}(n + 1) \)-ic can pass, or by two less than that number, a \( \frac{1}{2}(n - 1) \)-ic cannot be determined. Hence, when \( n \) is odd, the \( \frac{1}{2}n(n + 3) \) arbitrary double points cannot be the intersections of a \( \frac{1}{2}(n - 1) \)-ic and a \( \frac{1}{2}(n + 1) \)-ic. Still less can they be the intersections of any other curves into which the \( n \)-ic might break up. When \( n \) is even, we have to consider the danger of the \( n \)-ic breaking up into two \( \frac{1}{2}n \)-ics. The \( \frac{1}{2}n(n + 3) \) double points
are too many for the determination of a \( \frac{1}{n} \)-ic when \( n > 8 \). When \( n = 8 \), we have fourteen arbitrary double points and two more conditions. If those conditions be that the curve have two more double points (not given), then two coincident quartics would meet the requirements, but if the two remaining conditions are not given then, in general, the 8-ic remains proper.

When \( n = 6 \), the number of double points which can be taken arbitrarily is eight. Dr. Salmon says (loc. cit.) that, if the curve is required to have nine double points, then not so many as eight of them can be assumed arbitrarily, but, in a letter which I have just received from him he says that this line will be removed from the next edition of his work.

To resume, the number of double points which can be taken arbitrarily when \( n < 6 \) is \( \frac{1}{2} (n-1)(n-2) \), when \( n = 6 \) is \( \frac{1}{2} n(n+3) - 1 \),

\[
\begin{align*}
\text{" } n = 7 \text{ " } & \frac{1}{2} n(n+3), \\
\text{" } n = 8 \text{ " } & \begin{cases} \\
& \frac{1}{2} n(n+3) \text{ or} \\
& \frac{1}{2} n(n+3) - 1 \\
\end{cases}
\end{align*}
\]

as the curve is or is not required to have two more double points, and when \( n > 8 \) it is \( \frac{1}{2} n(n+3) \).

158. "Let two concentric and similarly placed ellipses, infinitely near each other, be described, the semi-axes of the inner being \( a \) and \( b \), and those of the outer \( a + da \), and \( b + da \); show that the minimum distance between their perimeters \( = 2\sqrt{(ab)da} - (a+b) \)."

**SOLUTION BY PROF. JOHNSON.**

The perpendicular upon a tangent to the ellipse is

\[
p = \sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha}
\]

where \( \alpha \) is the inclination of \( p \) to the axis of \( x \). The problem requires the minimum value of \( dp \) taken on the supposition that \( da = db \), and \( a \) constant. This is

\[
dp = \frac{a \cos^2 \alpha + b \sin^2 \alpha}{\sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha}} \, da.
\]

Regarding \( dp + da \) as a function of \( \alpha \) we find it to be a max. when \( \sin \alpha \times \cos \alpha = 0 \) and a min. when \( \tan^2 \alpha = a/b \); substituting in the value of \( dp \),

\[
dp = 2\sqrt{(ab)} \, da - (a+b).
\]

159. "The first of two casks contains \( \alpha \) gallons of wine" &c. [See p. 64.]

**SOLUTION BY E. B. SEITZ.**

Let \( u_{\alpha} \) be the number of gallons of wine in the second cask after \( n \) oper'ns. Hence \( a + c = u_{\alpha} \) be the wine in the first cask after the \( n \) operations, and \( u_{\alpha} + e(a+c-u_{\alpha})/(a+b) \) = the wine in the second cask after putting \( e \) gals.
into it from the first in the \((n + 1)\)th operation, and in the second part of the operation \(\varepsilon = (a + d + e)\) of the wine in the second cask is poured into the first; hence \((a + d) = (c + d + e)\) of the wine remains.

\[
\therefore \quad u_{n+1} = \left[ \frac{(c+d)(a+b-e)}{(a+b)(c+d+e)} \right] u_n = \frac{c(a+c)}{a+b} \left( \frac{c+d}{a+b} \right) . \quad (1)
\]

The integral of \((1)\) is

\[
u_n = C \left[ \frac{(c+d)(a+b-e)}{(a+b)(c+d+e)} \right]^n + \frac{(a+c)(c+d)}{a+b+c+d} . \quad (2)
\]

When \(n = 0\), \(u_0 = c\); \(\therefore C = c - [(a+c)(c+d)]/(a+b+c+d)\).

\[
u_n = c \left[ \frac{(c+d)(a+b-e)}{(a+b)(c+d+e)} \right]^n + \frac{(a+c)(c+d)}{a+b+c+d} \left\{ 1 - \frac{(c+d)(a+b-e)}{(a+b)(c+d+e)} \right\}^n . \]

160—161. [See page 64.]

**SOLUTION BY HENRY HEATON.**

160. Multiplying the given equation by \([y + f(x)]^{-n}\) we have

\[
y dy [y + f(x)]^{-n} + (P - Q y) [y + f(x)]^{-n} dx.
\]

If this is integrable

\[
\frac{d}{dx} \left[ \frac{y}{y + f(x)} \right]^n = \frac{d}{dx} \left[ \frac{P - Q y}{y + f(x)} \right]^n ; \quad \therefore - n y \frac{df(x)}{dx} (y + f(x))^{n-1} = - Q [y + f(x)]^{-n} - n (P + Q y) [y + f(x)]^{-n-1}.
\]

\[
\therefore \left( \frac{df(x)}{dx} + (n-1) Q \right) y = Q f(x) - n P = 0 . \quad \therefore f(x) = \frac{1}{n} \int Q dx,
\]

and \(P = - Q f(x) + n\).

161. The centers of the random circles must be on the surface of a circle whose radius is \(r\). Let \(O\) be the center of this circle, and let \(O'\) and \(O''\) be the centers of the random circles. Put \(OO' = x\) and \(O'O'' = y\).

If \(y < (r-x)\), \(O''\) must be in the circumference of a circle whose center is \(O'\) and whose radius is \(y\); but if \(y > (r-x)\), it will be in an arc of a circle, having the same center and radius, but terminated by the circumference of the circle \(O\). The length of this arc is \(2y \cos^{-1}[(x^2 + y^2 - r^2)/(2xy)]\).

The area common to two circles whose radius is \(r\) and the distance between whose centers is \(y\), is \(A = 2r^2 \cos^{-1}(y + 2r - \frac{1}{2} \sqrt{4r^2 - y^2})\), and the area's area

\[
= \frac{1}{r^3} \int_0^r \left[ \int_0^x 2xydy + \int_{r-x}^2 2y \cos^{-1} \left( \frac{x^2 + y^2 - r^2}{2xy} \right) dy \right] 2\pi y dx
\]

\[
= \frac{2}{r^3} \int_0^r A^2 ydy = r^2 \pi - \frac{16r^2}{3\pi}.
\]
ANSWER TO PROF. HALL'S QUERY (SEE P. 48) BY PROF. H. T. EDDY.

Take the well known Eulerian integral

\[
\int_0^\pi \sin^p \varphi \cos^q \varphi \, d\varphi = \frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{2 \Gamma\left(\frac{p+q+2}{2}\right)}; \quad p = \frac{1}{2}, \quad q = 0.
\]

\[\therefore \quad u = \frac{\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{1}{4}\right)}{2 \Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{1}{4}\right)} = \frac{\sqrt{\pi}}{2} \Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{1}{4}\right).
\]

Again, \[
\frac{[\Gamma(n)]^2}{\Gamma(n-m) \Gamma(n+m)} = \left(1 - \frac{m^2}{n^2}\right) \left(1 - \frac{m^2}{(n+1)^2}\right) \left(1 - \frac{m^2}{(n+2)^2}\right) \ldots .
\]

In this equation make \( m = \frac{1}{4} \), and, first, put \( n = \frac{1}{4} \) and then \( n = \frac{3}{4} \). Dividing the first result by the second we get

\[
\frac{\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{1}{4}\right) [\Gamma\left(\frac{1}{4}\right)]^2}{\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{1}{4}\right) [\Gamma\left(\frac{1}{4}\right)]^2} = \frac{[\Gamma\left(\frac{1}{4}\right)]^2}{2 [\Gamma\left(\frac{1}{4}\right)]^2}.
\]

\[\therefore \quad u = \left[\begin{array}{c}
\frac{\pi}{2} \frac{\pi}{2} \frac{\pi}{2} \frac{\pi}{2} \frac{\pi}{2} \frac{\pi}{2} \frac{\pi}{2} \ldots \\
\frac{\pi}{2} \frac{\pi}{2} \frac{\pi}{2} \frac{\pi}{2} \frac{\pi}{2} \frac{\pi}{2} \frac{\pi}{2} \ldots
\end{array}\right]^\frac{1}{2},
\]

which is in a form convenient for computation from a table of logarithms, and especially it is easy to obtain the logarithms of the fractions at the right of those given as they are simple tabular differences. Fifteen terms or perhaps less, are sufficient to obtain \( u = 1.198 \ldots \).

ANSWER TO MR. HEAL’S QUERY. (SEE PAGE 64.)

[T. P. STOWELL of Rochester, N. Y., writes: “Perhaps it would interest some of your readers to reprint a paper published in the Math. Repository (Vol. I, 2nd series) in 1806.” As our space will not permit the publication of the paper referred to, in full, we give the following extract, and subjoin the Construction of a polygon of 17 sides, sent by Mr. Stowell, and credited to Leybourn’s Math. Repository, 1818.]

“Mr. Gauss, of Brunswick, published at Leipsic, in 1801, a work called Disquisitiones Arithmeticae. In this work the author announces, that we may always inscribe a regular polygon of \(2^n + 1\) sides in a circle, when \(n\) is a whole number, and \(2^n + 1\) a prime number.

“Mr. Legendre has given to the French National Institute a demonstration of this very curious proposition in the case when the number of sides is 17. It is founded on these two lemmas:

Lemma I. Let \(a\) be the arch of a circle, \(m\) and \(n\) two whole numbers, then

\[2 \cos ma \cos na = \cos (m-n)a + \cos (m+n)a.
\]

Lemma II. Let \(a\) be the \(n\)th part of a whole circumference \(n\) being a whole number, then

\[\cos a + \cos 2a + \cos 3a + \ldots + \cos na = 0.\]
"To construct a regular polygon of seventeen sides in a circle.

Draw the radius \( CO \) at right-angles to the diameter \( AB \): On \( OC \) and \( OB \), take \( OQ \) equal to the half, and \( OD \) equal to the eighth part of the radius: Make \( DE \) and \( DF \) each equal to \( DQ \), and \( EG \) and \( FH \) respectively equal to \( EQ \) and \( FQ \); take \( OK \) a mean proportional between \( OH \) and \( OQ \), and through \( K \), draw \( KM \) parallel to \( AB \), meeting the semicircle described on \( OG \) in \( M \); draw \( MN \) parallel to \( OC \), cutting the given circle in \( N \)—the arc \( AN \) is the seventeenth part of the whole circumference."

---

**PROBLEMS.**

162. **SELECTED.**—Given

\[ x^3 + xy + y^3 = 37, \quad (1) \]
\[ x^3 + zx + z^3 = 49, \quad (2) \]
\[ y^3 + yz + z^3 = 61, \quad (3) \]

To find \( x \), \( y \) and \( z \) by quadratics.

163 **BY PROF. ORSON PRATT, SEN.**—Resolve the first member of the general cubic equation, \( x^3 + px^3 + qx = -r \), into three factors, such that, when their signs are changed, their sum shall equal \( p \); when their signs are unchanged the sum of their products, taken two and two, shall equal \( q \); and when their signs are unchanged, their continued product shall equal \( -r \). Or, in other words, find the forms of the three roots in terms of \( x \) and the coefficients.

164. **BY PROF. W. W. BEMAN, ANN ARBOR, MICH.**—From any two points to draw two lines which shall meet in the circumference of a given circle, and make equal angles with the tangent at the point of intersection.

165. **FROM BOOLE'S DIF. EQ'NS (by request).**—Of the system of dynamical equations,

\[ \frac{d^2x}{dt^2} + \frac{mx}{r^3} = 0, \quad \frac{d^3y}{dt^2} + \frac{my}{r^3} = 0, \quad \frac{d^3z}{dt^2} + \frac{mz}{r^3} = 0, \]

where \( r = \sqrt{x^2 + y^2 + z^2} \), seven first integrals are obtained of which it is subsequently found that five only are independent. How many final integrals can hence be deduced without proceeding to another integration.
166. BY PROF. W. P. CASEY, SAN FR. CAL.—If through a point $O$, within a given triangle $ABC$, three lines be drawn respectively parallel to the sides of the triangle; viz., $GE$ parallel to $BC$, $FH$ to $AB$ and $DT$ to $AC$, and there is given, $GO \times OE + FO \times OH + DO \times OT$; to find the locus of $O$ by plane geometry.

167. BY PROF. C. M. WOODWARD.—A solid sphere rolls down a trough formed by two planes which make with each other an angle $2\phi$. Find an expression for the time when the inclination of the trough to the horizon is $\theta$.

168. (SELECTED) BY PROF. H. T. EDDY.—Find the general value of

$$u = \int \int \frac{dxdy}{\sqrt{(1 + x^2 + y^2)^3}}$$

and show that when the limits of $x$ and $y$ are 0 and 1, $u = 0.5 +$.

169. BY PROF. JOHNSON.—Base balls are covered by sewing together two dumb-bell shaped pieces of leather. Determine the shape of the pieces so as to reduce the distortion in fitting them to the spherical surface to a minimum.

QUERY. BY DR. N. R. OLIVER.—How is the Rule, given at page 44, Gillespie’s Land-surveying, (5th edition, New York, 1857,) demonstrated?

QUERY. BY GEORGE LILLEY, KEWANEE, ILL.—It is stated that Prof. Paolo Gorini, of Lodi, in Italy, has recently succeeded in demonstrating Peter De Fermat’s famous theorem. What is the demonstration?

NOTICES OF PUBLICATIONS RECEIVED.

NOTE ON THE SENSATION OF COLORS. By C. S. PEIRCE. This paper is a mathematical discussion of the change produced in the sensation of color by varying the intensity of the light. The paper is a contribution to the Amer. Journal of Science for April, 1877, and will be continued in the May No. of that Journal.

ON THE ATMOSPHERES OF THE SUN AND PLANETS. By DAVID TROWBRIDGE, A. M. This is a pamphlet of seven pages, 8vo., and is an attempt to estimate the probable volume, density and height of the atmospheres that are supposed to surround those bodies. The pamphlet embodies a paper read before the Amer. Phil. Soc., Nov. 3, 1876, and published in the Proceedings for that year.

THE MATHEMATICAL VISITOR, NO. 1, EDITED AND PUBLISHED BY ARTEMAS MARTIN. The Visitor is devoted mainly to the solution of questions, and the editor announces that it will be published annually; No. 2 will appear about the first of Jan. 1878. No. 1 is a 4to of eleven pages; price 20 cts. No. 2 will contain 32 pages and its price will be 50 cts. Address, Lock Box 11, Erie, Pa.
EMPIRICAL FORMULA FOR THE VOLUME OF ATMOSPHERIC AIR.

BY G. W. HILL.

The formula of Mariotte and Gay-Lussac is generally employed, in the laboratory, to reduce volumes, observed under one tension and temperature, to those which would have place under other tensions and temperatures. But Regnault, about 1845, made several series of experiments, which, if they may be relied upon, establish marked deviations from this formula. These experiments are detailed in the Mémoires de l'Académie des Sciences de Paris, Tom. XXI. I propose to investigate a modification of the formula, the introduction of which makes it possible to satisfy nearly these experiments.

Let $T$ denote the temperature, here always expressed in degrees of the centigrade scale; $P$ the tension or pressure, measured by the altitude, in metres, of a column of mercury; it is capable of supporting; the mercury being at the temperature 0 and under the action of gravity which obtains at Regnault's laboratory; and let $V$ denote the volume. Then, for any given mass of air, these three quantities are so connected that if any two of them are assigned the remaining third is immediately determined. That is, we must have

function $(V, P, T) = 0$,

or, solved with respect to $V$,

$V = \text{function} (P, T)$.

But the mode, in which $T$ is to be measured, is arbitrary, and we may take atmospheric air as the thermometric substance, and assume that $T$ increases, in direct proportion, as the volume, under constant pressure, increases. This gives

$V = F(P) + f(P).T$. 
It is here taken for granted that, whatever may be the density of the air inclosed in the thermometer, its indications will be the same. It is true that the usual custom of experimenters has been to measure temperatures by the augmentation of tensions under constant volume; but, when Mariotte's law holds, this gives results identical with those obtained by the former method. In this case we shall have to write the equation

\[ P = F(V) + f(V) T. \]

But the first equation seems preferable.

Now since, for any given constant temperature, the volume ought to be a function of the tension similar to what it is at any other temperature, it follows that, if \( F(P) \) is supposed to consist of a series of terms, each of the form \( KP^k \), where \( K \) and \( k \) are constants, so that we may write

\[ F(P) = \Sigma K P^k, \]

then we ought to have

\[ f(P) = \Sigma K_1 P^k, \]

where \( K_1 \) denotes a constant, in general, different from \( K \). Thus we should have

\[ V = \Sigma [K + K_1 T] P^k. \]

The formula of Mariotte and Gay-Lussac assumes that \( F(P) \) and \( f(P) \) contain each only one term, in which \( k = -1 \). But Regnault's experiments having shown the insufficiency of this, it is in order to see whether agreement between theory and observation cannot be brought about by annexing to \( V \) an additional term, for which \( k \) has a value different from \(-1\). Thus let us suppose that

\[ V = [K + K_1 T] P^{-1} + [K' + K_1' T] P^{k-1} \]

\[ = \frac{K + K' P^k}{P} + \frac{K_1 + K_1' P^k}{P} T. \]

As \( V \) contains a factor, which is directly proportional to the mass of air considered and inversely as the unit assumed for the measurement of volumes, we prefer to write the preceding equations thus

\[ V = K \left[ \frac{1+a' T}{P} + \frac{a+ a'' T P^k}{P} \right] \]

\[ = K \left[ \frac{1+a P^k}{P} + \frac{a'+ a'' P^k}{P} T \right]. \]

When the temperature is constant, the volumes are represented by the formula

\[ V = K \frac{1+a P^k}{P}, \]
that is, the result from Mariotte's law must be multiplied by the factor $1 + \alpha P^\beta$, which differs but little from unity. $\alpha$ is a small constant which measures the amplitude of the deviations from this law; while $\beta$ is a constant exponent, which is so chosen that the more or less rapid variation of the deviations, in passing from one tension to another, may be represented as well as possible. It is evident that, in this manner, we get the utmost advantage that can be derived from the addition of a single term to $V$.

The experiments of Regnault may be divided into two classes; first, those where, the temperature remaining nearly constant, the volumes of the same mass of air, under different pressures, were observed; second, those where, the volumes remaining nearly the same, the tensions were observed at the temperature of freezing and boiling water. It is obvious that experiments of these two kinds extended over a sufficient range of tension would afford the data requisite for obtaining the values of the four constants $\alpha$, $\alpha'$, $\alpha''$ and $\beta$ which enter into our adopted formula.

The experiments of the first class are enumerated at pp. 374—379 of the volume quoted above. As the temperature is nearly the same for all, we assume that they have been made at the average of all the noted temperatures which is $40.747$.

To save labor we may take the average of the observed volumes and tensions when they are nearly alike. In this way Regnault's 66 experiments are reduced to the 23 given in the following table. It may be noted that $V$ is here expressed by the number of grammes of mercury required to fill the volume. The column containing $\log (PV)$ exhibits the deviation from Mariotte's law; did this law exactly hold, the numbers in this column would be identical for each series. It will be noted that, in general, they diminish with increasing pressures. The volumes being supposed to be represented by the equation

$$V = K \frac{1 + \alpha P^\beta}{P},$$

a preliminary investigation has given the approximate values

$$\alpha = -0.0024337, \quad \beta = 0.645.$$

With these have been computed the values of the expressions which stand at the head of the two last columns of the table, and which serve to obtain the coefficients of the equations of condition to be given presently.

As the mass of air operated on was different in each series of experiments, $K$ will have 9 different values; it can, however, be eliminated. Taking the common logarithms of each member of the equation last given,

$$\log K + \log (1 + \alpha P) = \beta \log (PV).$$
To reduce the matter within the treatment of the method of least squares, it will be necessary to make some assumption regarding the probable errors of the observed $P$ and $V$. We will, for convenience, suppose that they are such that the function $\log (PV)$ has a probable error equal for all the observations; an assumption somewhat precarious, it is true, but it seems that we cannot easily do better.

Let the small corrections, which it is necessary to apply to the approximate values of $\log K$, $a$ and $\beta$, be denoted by $\delta \log K$, $\delta a$ and $\delta \beta$, and let us put

$$\delta \log K = z, \quad M \delta a = y, \quad a \delta \beta = z,$$

where $M$ denotes the modulus of common logarithms. $\delta \log (PV)$ being the excess of observed over calculated $\log (PV)$, we shall have the equation of condition

$$x + \frac{P^\beta}{1+aP^\beta} y + \frac{P^\beta}{1+aP^\beta} \log Pz = \delta \log (PV).$$
A little consideration will show that \( z \) will be eliminated by taking the difference of every two equations of condition arising from the same series, and attributing the weight \( w w' + \Sigma \omega \) to the resulting equation, \( w \) and \( w' \) denoting the weights of the equations whose difference is taken, and \( \Sigma \omega \) the sum of the weights of all the equations in the series. Since the coefficients of \( y \) and \( z \) in the equations, are all positive and nearly proportional, it will be advantageous to adopt a new unknown \( u \), such that

\[
y = u - \frac{3}{4}z.
\]

Then the equations, with the weights that ought to be attributed them, are

<table>
<thead>
<tr>
<th>Series</th>
<th>Equation</th>
<th>Weight, ( w )</th>
</tr>
</thead>
</table>
| 1st    | \[
0.4653u - 0.4490z = -0.000106
\] | 2 |
| 2nd    | \[
0.9168 - 0.4685 = -0.000193
\] | 4 |
| 3rd    | \[
1.6951 - 0.6869 = + 0.000255
\] | 4 |
| 4th    | \[
0.7783 - 0.2184 = + 0.000448
\] | 4 |
| 5th    | \[
0.9057 - 0.4709 = -0.000252
\] | 2 |
| 6th    | \[
1.3991 - 0.3113 = + 0.000076
\] | 1.2 |
| 7th    | \[
1.4398 - 0.3061 = + 0.000038
\] | 1.2 |
| 8th    | \[
2.7461 - 0.2764 = + 0.000432
\] | 4 |
| 9th    | \[
1.3063 + 0.0296 = + 0.000394
\] | 4 |
|        | \[
1.9603 - 0.0183 = + 0.000002
\] | 1.2 |
|        | \[
3.3290 + 0.2605 = -0.000407
\] | 0.3 |
|        | \[
3.3989 + 0.2800 = + 0.000104
\] | 0.6 |
|        | \[
1.3687 + 0.2787 = -0.000409
\] | 0.4 |
|        | \[
1.4386 + 0.2982 = + 0.000102
\] | 0.8 |
|        | \[
0.0699 + 0.0195 = + 0.000011
\] | 0.2 |
|        | \[
1.8901 - 0.0642 = -0.000059
\] | 0.8 |
|        | \[
3.5495 + 0.2635 = -0.000117
\] | 0.4 |
|        | \[
1.6594 + 0.3276 = -0.000058
\] | 0.4 |
|        | \[
2.4166 + 0.3099 = -0.000164
\] | 1.5 |

The derived normal equations are

\[
58.672u - 0.0790z = -0.0005252,
\]

\[
- 0.079u + 2.4453z = -0.0000157.
\]

Whence

\[
u = -0.000008962, \quad z = -0.000004707, \quad y = +0.000003216.
\]

\[
\delta a = +0.0000051, \quad \delta \beta = +0.000276.
\]

Applying these corrections to the approximate values of \( a \) and \( \beta \), we get the final values

\[
a = -0.0024286, \quad \beta = +0.64776.
\]

How well the experiments are represented by the formula, with these values of the constants, will best be seen from the following comparison of the values of \( V_0P_0 + V_1P_1 \) given by Regnault and those computed from the formula.
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1.001414</td>
<td>1.001133</td>
<td>+ 281</td>
<td>1.005437</td>
<td>1.006694</td>
<td>- 1257</td>
</tr>
<tr>
<td>1.001448</td>
<td>1.001132</td>
<td>+ 316</td>
<td>1.005703</td>
<td>1.006694</td>
<td>- 991</td>
</tr>
<tr>
<td>1.001224</td>
<td>1.001133</td>
<td>+ 91</td>
<td>1.004286</td>
<td>1.004777</td>
<td>- 491</td>
</tr>
<tr>
<td>1.001421</td>
<td>1.001133</td>
<td>+ 288</td>
<td>1.004512</td>
<td>1.004770</td>
<td>- 258</td>
</tr>
<tr>
<td>1.002765</td>
<td>1.002233</td>
<td>+ 532</td>
<td>1.004599</td>
<td>1.004779</td>
<td>- 180</td>
</tr>
<tr>
<td>1.002759</td>
<td>1.002234</td>
<td>+ 525</td>
<td>1.004580</td>
<td>1.004771</td>
<td>- 191</td>
</tr>
<tr>
<td>1.002503</td>
<td>1.002236</td>
<td>+ 267</td>
<td>1.008536</td>
<td>1.008106</td>
<td>+ 430</td>
</tr>
<tr>
<td>1.003539</td>
<td>1.004134</td>
<td>- 595</td>
<td>1.008813</td>
<td>1.008108</td>
<td>+ 705</td>
</tr>
<tr>
<td>1.003452</td>
<td>1.004133</td>
<td>- 681</td>
<td>1.008016</td>
<td>1.008286</td>
<td>- 270</td>
</tr>
<tr>
<td>1.003309</td>
<td>1.004133</td>
<td>- 824</td>
<td>1.008064</td>
<td>1.008269</td>
<td>- 205</td>
</tr>
<tr>
<td>1.002709</td>
<td>1.002209</td>
<td>+ 500</td>
<td>1.007980</td>
<td>1.008288</td>
<td>- 308</td>
</tr>
<tr>
<td>1.002724</td>
<td>1.002207</td>
<td>+ 517</td>
<td>1.004611</td>
<td>1.004601</td>
<td>+ 10</td>
</tr>
<tr>
<td>1.002713</td>
<td>1.002206</td>
<td>+ 507</td>
<td>1.004752</td>
<td>1.004601</td>
<td>+ 151</td>
</tr>
<tr>
<td>1.002528</td>
<td>1.002211</td>
<td>+ 317</td>
<td>1.008330</td>
<td>1.008648</td>
<td>+ 282</td>
</tr>
<tr>
<td>1.002898</td>
<td>1.002203</td>
<td>+ 695</td>
<td>1.008755</td>
<td>1.008642</td>
<td>+ 113</td>
</tr>
<tr>
<td>1.002762</td>
<td>1.002203</td>
<td>+ 559</td>
<td>1.006366</td>
<td>1.005876</td>
<td>+ 490</td>
</tr>
<tr>
<td>1.003253</td>
<td>1.003417</td>
<td>- 164</td>
<td>1.006132</td>
<td>1.005880</td>
<td>+ 252</td>
</tr>
<tr>
<td>1.003090</td>
<td>1.003411</td>
<td>- 321</td>
<td>1.006010</td>
<td>1.005869</td>
<td>+ 141</td>
</tr>
<tr>
<td>1.003302</td>
<td>1.003407</td>
<td>- 105</td>
<td>1.006346</td>
<td>1.005878</td>
<td>+ 468</td>
</tr>
<tr>
<td>1.003336</td>
<td>1.003506</td>
<td>- 170</td>
<td>1.005619</td>
<td>1.005738</td>
<td>- 121</td>
</tr>
<tr>
<td>1.003495</td>
<td>1.003508</td>
<td>- 13</td>
<td>1.005622</td>
<td>1.005736</td>
<td>- 114</td>
</tr>
<tr>
<td>1.003335</td>
<td>1.003508</td>
<td>- 173</td>
<td>1.005902</td>
<td>1.005832</td>
<td>+ 70</td>
</tr>
<tr>
<td>1.003448</td>
<td>1.003509</td>
<td>- 61</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

It will be seen that the differences, in the extreme cases, amount to a fourth part of the observed deviation from the law of Mariotte. Moreover it is plain that some cause, which varied from series to series, has operated to vitiate these experiments, since it is possible to determine \( \alpha \) and \( \beta \) so that any two series are well represented, but not possible when all the series are included in the investigation. It may be noted also that the experiments in which the original volume was reduced to one third are not in general concordant with those where the reduction was to one half.

That these discrepancies are unavoidable will be evident from the following exposition. Let us put

\[
\text{com. log } (PV) = F(P).
\]

The observations of Regnault may be resumed in the following 9 results, all formed by combining tolerably concordant data,

1. \( F(1.476) - F(0.739) = 0.000598 \),
2. \( F(4.168) - F(2.091) = 0.001181 \),
3. \( F(6.350) - F(2.112) = 0.001526 \),
4. \( F(8.292) - F(4.182) = 0.001437 \),
These are the data actually furnished by Regnault for the determination of the function $F(P)$. Employing the graphical method, we endeavor to construct the curve whose equation is $y = F(x)$. One of the special values of $F(x)$, as $F(0.739)$, may be taken arbitrarily, and then the value of $F(1.476)$ becomes known. This premised, we see that each of the 9 equations furnishes the length, direction and abscissa of the extremities of a chord, of the sought curve. Placing the chord, corresponding to the first equation, arbitrarily, and drawing the others on any part of the paper, but with the correct direction, and correct abscissa of their extremities, we endeavor, by imparting a motion to all their points parallel to the axis of $y$, to make them fall into line as the chords of the same continuous curve. We find that if we take 1, 2, 4, 6 and 7, they can be made to indicate a tolerably continuous curve; but then 3, 5, 8 and 9 are not satisfied.

Again, from this graphical process, we see that there cannot be much variation of curvature between the extremities of each chord, and hence the tangent to the curve, corresponding to the abscissa, which is the mean of the abscissae of the extremities, ought to be, very approximately, parallel to the chord; or in other terms

$$\frac{d}{dx} F\left(\frac{x_1 + x_2}{2}\right) = \frac{F(x_1) - F(x_2)}{x_1 - x_0}.$$  

This gives the following values of $\frac{dy}{dx}$:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\frac{dy}{dx}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.108</td>
</tr>
<tr>
<td>2</td>
<td>3.130</td>
</tr>
<tr>
<td>3</td>
<td>4.231</td>
</tr>
<tr>
<td>4</td>
<td>6.237</td>
</tr>
<tr>
<td>5</td>
<td>8.600</td>
</tr>
<tr>
<td>6</td>
<td>9.840</td>
</tr>
<tr>
<td>7</td>
<td>12.428</td>
</tr>
<tr>
<td>8</td>
<td>13.940</td>
</tr>
<tr>
<td>9</td>
<td>16.265</td>
</tr>
<tr>
<td></td>
<td>+0.0008113</td>
</tr>
<tr>
<td></td>
<td>0.0005686</td>
</tr>
<tr>
<td></td>
<td>0.0003770</td>
</tr>
<tr>
<td></td>
<td>0.0003497</td>
</tr>
<tr>
<td></td>
<td>0.0002788</td>
</tr>
<tr>
<td></td>
<td>0.0003131</td>
</tr>
<tr>
<td></td>
<td>0.0002948</td>
</tr>
<tr>
<td></td>
<td>0.0002914</td>
</tr>
<tr>
<td></td>
<td>+0.0002594</td>
</tr>
</tbody>
</table>
From the general course of these values of \(dy + dx\) it may be gathered that this function, at first, diminishes rapidly, afterwards more slowly, and then tends, with higher values of \(x\), to become nearly constant. But while this is the conclusion from the tout ensemble, a comparison of some of the values contradicts it. Thus from 1, 2 and 3, while \(dy + dx\) diminishes 0.0002427 in an interval 2.0 in \(x\), it afterwards diminishes 0.0001916 in an interval 1.1 of \(x\). All attempts then to represent these data by a curve, without singular points, must, evidently, show large errors.

For the discussion of the second class of experiments, let us assume that \(\alpha\) has the significarion we have given it in the general formula for \(V\). Then the volume remaining the same, if \(P_0\) and \(P_1\) denote the tensions observed respectively at 0° and 100°, we have

\[
\frac{P_1}{P_0} = \frac{1 + 100\alpha' + (\alpha + 100\alpha'')P_1}{1 + \alpha P_0},
\]

\(P_1 / P_0\) is the quantity Regnault has designated by 1 + 100\(\alpha\), let us denote it by \(A\); then if, for convenience, we put

\[
\gamma = 1 + 100\alpha', \quad \gamma' = \alpha + 100\alpha'',
\]

each determined value of \(A\) will give the equation of condition

\[
\gamma + P_1^\beta \gamma' = A + AP_0^\beta \alpha.
\]

The following are Regnault’s determinations of \(A\) augmented, in general, by 0.00018, for the reason that we adopt the mean coefficient 0.00018153 for the expansion of mercury between 0° and 100°, found by this experimenter, instead of the value \(\pi_{\varphi_{\delta}}\) used by him, (See Note, p. 31 of the vol.); the last column contains the page of the volume, where the experiments may be found.

<table>
<thead>
<tr>
<th>(P_0)</th>
<th>(P_1)</th>
<th>(A)</th>
<th>No. Obs.</th>
<th>Obs.-Cal.</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°.110</td>
<td>0°.149</td>
<td>1.36500</td>
<td>10</td>
<td>-0.00012</td>
<td>99</td>
</tr>
<tr>
<td>0.174</td>
<td>0.237</td>
<td>1.36531</td>
<td>3</td>
<td>-0.00004</td>
<td>99</td>
</tr>
<tr>
<td>0.266</td>
<td>0.362</td>
<td>1.36560</td>
<td>2</td>
<td>-0.00003</td>
<td>99</td>
</tr>
<tr>
<td>0.375</td>
<td>0.510</td>
<td>1.36598</td>
<td>4</td>
<td>+0.00005</td>
<td>99</td>
</tr>
<tr>
<td>0.548</td>
<td>0.746</td>
<td>1.36673</td>
<td>3</td>
<td>+0.00038</td>
<td>57</td>
</tr>
<tr>
<td>0.756</td>
<td>0.7535</td>
<td>1.36724</td>
<td>4</td>
<td>+0.00035</td>
<td>66</td>
</tr>
<tr>
<td>0.557</td>
<td>0.754</td>
<td>1.36651</td>
<td>18</td>
<td>+0.00014</td>
<td>43</td>
</tr>
<tr>
<td>0.656</td>
<td>0.757</td>
<td>1.36641</td>
<td>14</td>
<td>-0.00022</td>
<td>33</td>
</tr>
<tr>
<td>0.747</td>
<td>1.016</td>
<td>1.36663</td>
<td>3</td>
<td>-0.00014</td>
<td>58</td>
</tr>
<tr>
<td>0.771</td>
<td>1.049</td>
<td>1.36696</td>
<td>11</td>
<td>+0.00014</td>
<td>51</td>
</tr>
<tr>
<td>1.678</td>
<td>2.286</td>
<td>1.36778</td>
<td>2</td>
<td>-0.00059</td>
<td>109</td>
</tr>
<tr>
<td>1.693</td>
<td>2.306</td>
<td>1.3818</td>
<td>4</td>
<td>-0.00021</td>
<td>109</td>
</tr>
<tr>
<td>2.526</td>
<td>2.517</td>
<td>1.36962</td>
<td>2</td>
<td>-0.00018</td>
<td>114</td>
</tr>
<tr>
<td>2.622</td>
<td>2.614</td>
<td>1.36982</td>
<td>2</td>
<td>-0.00011</td>
<td>114</td>
</tr>
<tr>
<td>2.144</td>
<td>2.924</td>
<td>1.36912</td>
<td>2</td>
<td>+0.00007</td>
<td>109</td>
</tr>
<tr>
<td>3.656</td>
<td>4.992</td>
<td>1.37109</td>
<td>4</td>
<td>+0.00031</td>
<td>109</td>
</tr>
</tbody>
</table>
Adopting, for convenience, as an unknown in the place of \( \gamma \),
\[
x = \gamma + \gamma' - 1.367,
\]
we have the following equations, to each of which we attribute a weight equal to a tenth of the number of experiments it is founded upon:
\[
\begin{align*}
x &= 0.7086\gamma' - 0.3268a = - 0.00200 \\
x &= 0.6064\gamma' - 0.4398a = - 0.00169 \\
x &= 0.4821\gamma' - 0.5790a = - 0.00140 \\
x &= 0.3534\gamma' - 0.7237a = - 0.00102 \\
x &= 0.1728\gamma' - 0.9258a = - 0.00027 \\
x &= 0.1674\gamma' - 1.1410a = + 0.00024 \\
x &= 0.1670\gamma' - 0.9354a = - 0.00049 \\
x &= 0.1650\gamma' - 1.040a = - 0.00059 \\
x &= 0.0105\gamma' - 1.131a = - 0.00037 \\
x &= 0.0317\gamma' - 1.155a = - 0.00004 \\
x &= 0.708 \gamma' - 1.913a = + 0.00078 \\
x &= 0.718 \gamma' - 1.924a = + 0.00118 \\
x &= 0.818 \gamma' - 2.496a = + 0.00262 \\
x &= 0.863 \gamma' - 2.558a = + 0.00282 \\
x &= 1.004 \gamma' - 2.243a = + 0.00212 \\
x &= 1.834 \gamma' - 3.175a = + 0.00409
\end{align*}
\]

The derived normal equations, for determining \( x \) and \( \gamma' \), are
\[
x - 0.0047\gamma' - 1.162a = - 0.000144, \\
- 0.0415x + 2.9547\gamma' - 3.373a = + 0.007074,
\]
whence
\[
\begin{align*}
x &= - 0.000133 + 1.168a, \\
\gamma' &= + 0.002392 + 1.158a,
\end{align*}
\]
and
\[
\begin{align*}
a' &= + 0.00364475 + 0.00010a, \\
a'' &= + 0.00002392 + 0.00158a.
\end{align*}
\]

The equation which determines \( a \) has already been obtained from the discussion of the first class of experiments; it is
\[
\frac{a + 4.747a''}{1 + 4.747a'} = - 0.0024286.
\]

The last three equations being solved, we gather that the volume of any mass of air is represented by the formula
\[ V = \frac{K}{P} \left[ 1 + aP^\beta + (a' + a''P^\beta)T \right], \]
in which
\[ a = -0.002565, \quad a' = +0.0036445, \]
\[ a'' = +0.00001987, \quad \beta = 0.64776. \]

How well the second class of experiments is satisfied by this formula may be seen from the numbers in the column headed Obs.-Cal.

If we have \( T = \frac{0.002565}{0.00001987} = 129.1 \), \( V \) takes the form
\[ V = \frac{K}{P}. \]

Hence we have the noteworthy result that:

*About the temperature 130°, air follows quite exactly the law of Mariotte.*

For the following temperatures and pressures the volume vanishes:

<table>
<thead>
<tr>
<th>( T )</th>
<th>( P )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0.49</td>
</tr>
<tr>
<td>-50</td>
<td>4420 .13</td>
</tr>
<tr>
<td>-100</td>
<td>2048 .00</td>
</tr>
<tr>
<td>-150</td>
<td>896 .26</td>
</tr>
<tr>
<td>-200</td>
<td>314 .23</td>
</tr>
</tbody>
</table>

These numbers may be regarded as indications of the magnitude of pressure necessary for the condensation of air. The table is in accordance with the well-known fact that reduction of temperature facilitates condensation.

A table is given below which will be found useful in the application of the formula. It contains the functions \( \log(1 + aP^\beta) \) and \( \frac{a' + a''P^\beta}{1 + aP^\beta} \), the latter being the coefficient of expansion under a constant pressure.

As an example let us suppose that the volume of a mass of air has been observed under the pressure 2 atm and the temperature 20°; it is required to find the factor necessary for reducing it to the pressure 0 atm and temperature 0°. From the table we get 3.07084. By employing the ordinary formula with the coefficient 0.003665 of expansion, there is obtained 3.06482, which differs from the preceding by about a 525th part.

Rigorously observations of pressure made in localities having an intensity of gravity different from that which prevails at Regnault's laboratory ought to be multiplied by the ratio of the former to the latter. The latitude of Regnault's laboratory is stated at 48° 50' 14'', the altitude above sea level at 60 m, and the intensity of gravity at 9 = 8096.
-107-

<table>
<thead>
<tr>
<th>P. log(1 + aP^\beta)</th>
<th>Coeff. of Exp.</th>
<th>P. log(1 + aP^\beta)</th>
<th>Coeff. of Exp.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.000000</td>
<td>0.036445</td>
<td>0.00037485</td>
</tr>
<tr>
<td>0.1</td>
<td>9.999749</td>
<td>36511</td>
<td>9.996053</td>
</tr>
<tr>
<td>0.2</td>
<td>9.999607</td>
<td>36548</td>
<td>9.995872</td>
</tr>
<tr>
<td>0.3</td>
<td>9.999489</td>
<td>36579</td>
<td>9.995695</td>
</tr>
<tr>
<td>0.4</td>
<td>9.999384</td>
<td>36607</td>
<td>9.995521</td>
</tr>
<tr>
<td>0.5</td>
<td>9.999288</td>
<td>36632</td>
<td>9.995352</td>
</tr>
<tr>
<td>0.6</td>
<td>9.999199</td>
<td>36652</td>
<td>9.995185</td>
</tr>
<tr>
<td>0.7</td>
<td>9.999115</td>
<td>36678</td>
<td>9.995021</td>
</tr>
<tr>
<td>0.8</td>
<td>9.999035</td>
<td>36698</td>
<td>9.994860</td>
</tr>
<tr>
<td>0.9</td>
<td>9.998958</td>
<td>36719</td>
<td>9.994702</td>
</tr>
<tr>
<td>1.0</td>
<td>9.998885</td>
<td>36738</td>
<td>9.994546</td>
</tr>
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ON ADJUSTMENT FORMULAS.

BY E. L. DE FOREST, M. A.

(Continued from page 86.)

Three separate tests of good adjustment have been proposed by the present writer, for which see Smithsonian Report of 1873, p. 333, and a more recent essay in pamphlet form; (Interpolation and Adjustment of Series, New Haven, 1876). The first of these tests is, that if we denote by \( v \) the residual error, found by subtracting an adjusted term from the corresponding
given one, and denote by $\varepsilon_1$ the mean error of the given term, then forming the values of $(v + \varepsilon_1)^2$ for all the terms of the series, we ought to have, denoting the whole number of terms by $N$,

$$\frac{1}{N} \sum \left( \frac{v}{\varepsilon_1} \right)^2 = 1 \pm 0.6745 \sqrt{\frac{2}{N}} \ldots \ldots (9)$$

that is to say, the arithmetical mean of all the values of $(v + \varepsilon_1)^2$ ought to be approximately equal to unity, and ought not to differ from it by more than its probable error, which is

$$0.6745 \sqrt{\frac{2}{N}}.$$

The several values of $\varepsilon_1$ are supposed to be determined from the nature of the observations, each given term being the mean result of a number of observations. The values of $\varepsilon_1$ will often vary for different terms of the series, not merely in inverse ratio to the square root of the number of observations made upon each term, but also as some function of the term itself, so that the mean error of one term cannot always be inferred from that of another term solely by comparing the numbers of observations made upon each. An illustration of this is furnished by formula (96), in the Smithsonian Report of 1873, p. 334.

There is an even chance that the true series, if we had it, would satisfy the test of good adjustment just described. If, therefore, we find that our adjusted series does satisfy it, we may consider that a good approximation to the truth has been reached, and perhaps as good as the nature of the case will permit. If the mean of $(v + \varepsilon_1)^2$ falls below the lower limit, that is falls short of unity by more than its probable error, we must infer that the residuals $v$ are too small, and that the series probably has not been smoothed out quite enough, while if it goes beyond the upper limit, so as to exceed unity by more than its probable error, the inference will be that it has been smoothed too much. It thus becomes interesting to see whether we cannot find some rule to guide us in the choice of a formula which will probably make an adjustment satisfying this test. Let us inquire then, what the most probable value of the arithmetical mean of $(v + \varepsilon_1)^2$ will be, for any given adjustment formula used.

Assuming that all the terms of the given series are of equal weight, let $n$ be a number such that the probable error, or deviation from zero, of the arithmetical mean of any $n$ consecutive values of $v$ shall be equal to the probable error of an adjusted term, so that we have

$$\varepsilon' = \frac{\varepsilon}{\sqrt{n}} \ldots \ldots \ldots \ldots \ldots (10)$$
The probable error of any one of these \( n \) terms will be approximately

\[
e = .6745 \sqrt{\frac{\sum (v^2)}{n-1}}.
\]

If we denote by \( v_1 \) the true error, or remainder after subtracting the true value of a term from its given or observed value, we shall also have

\[
e = .6745 \sqrt{\frac{\sum (v_1^2)}{n}}.
\]

Placing these two values of \( e \) equal to each other, and giving to \( n \) its value from (10), we get

\[
\frac{\sum (v^2)}{\sum (v_1^2)} = 1 - \left( \frac{e'}{e} \right)^2.
\]

But the most probable ratio of the two sums in the first member will evidently be unchanged if we extend the summation throughout the whole series of \( N \) terms, and dividing both numerator and denominator by \( e_1^2 \), and also by \( N \), we shall have

\[
\frac{1}{N} \sum \left( \frac{v}{e_1} \right)^2 = 1 - \left( \frac{e'}{e} \right)^2.
\]

Now in the first member the most probable value of the denominator is unity, because every value of \( v_1 \) is a true error in a system whose mean error is unity, and the square of the mean error is the mean of the squares of all the true errors. Hence we have, as the most probable value of the arithmetical mean of \( (v+\varepsilon_1)^2 \)

\[
\frac{1}{N} \sum \left( \frac{v}{e_1} \right)^2 = 1 - \left( \frac{e'}{e} \right)^2.
\]

It thus appears that this mean value will most probably be less than unity by the square of the error-ratio due to the adjustment formula employed, though it may and probably will exceed or fall short of this value in particular cases.* We have reached this result by assuming that the weights of the given terms are all equal, yet it will hold true approximately if they are unequal, for the probable value of \( v \) will vary in nearly the same ratio as \( e_1 \) does.

Since the proposed test of good adjustment requires that the mean of \( (v+\varepsilon_1)^2 \) shall not fall short of unity by more than its probable error, which is

\[
.6745 \sqrt{\frac{2}{N}}
\]

*Its probable error, or probable deviation from the most probable value, will be

\[
\pm .6745 \left\{ 1 - \left( \frac{e'}{e} \right)^2 \right\} \sqrt{\frac{2}{N}}
\]

Compare Interpolation and Adjustment of Series, p. 10.
if the adjusted series is the true one, it follows that we ought to choose an
adjustment formula whose error-ratio \( \epsilon' / \epsilon \) is at any rate small enough to
satisfy the equation

\[
1 - \left( \frac{\epsilon'}{\epsilon} \right)^2 = 1 - 0.6745\sqrt{\frac{2}{N}}
\]

that is, we should have

\[
\frac{\epsilon'}{\epsilon} \leq \sqrt{(0.6745 \sqrt{\frac{2}{N})}} \quad \ldots \ldots \ldots \quad (12)
\]

which may be written

\[
\frac{\epsilon'}{\epsilon} \leq \frac{0.9767}{\sqrt{N}}.
\]

If we wish to adjust a series of 50 terms, we cannot consider that the chan-
ces are in favor of the test being satisfied, unless we use an adjustment for-
uala whose error-ratio does not exceed

\[
\frac{0.9767}{\sqrt{50}} = 0.367,
\]

so that we should choose a formula of at least as many as 17 terms in Table
A., or 23 in Table B., or 19 in Table C. On the other hand, formula (12) gives

\[
N \leq \frac{2.(0.6745)^2}{\left( \frac{\epsilon'}{\epsilon} \right)^4} \quad \ldots \ldots \ldots \quad (13)
\]

that is to say

\[
N \leq \frac{0.9099}{(\epsilon'/\epsilon)^4}.
\]

Assigning to \( \epsilon' = \epsilon \) the successive values \( 0.300, \quad 0.345, \quad 0.311, \)
we find for \( N \) the values \( 112, \quad 64, \quad 97, \)
showing that while a series of as many as 112 terms can probably be ad-
justed by the 25-term formula of Table A. so as to meet the test, the long-
est formula of Table B. would probably succeed only for a series of 64 terms
or less, and that of Table C. for one of 97 terms or less.

The conclusion we come to from this investigation is, that it will be
advisable in most cases to use the longest adjustment formulas given in the
tables, provided that this can be done without violating the condition that
any \( 2m + 1 \) included terms must not deviate greatly from the form of a se-
ries of the third or any lower order.

As the first \( m \) and last \( m \) terms of the series cannot be reached directly by
the formula, the series should be graphically extended by \( m \) terms at both
ends, first plotting the observations on paper as ordinates, and then exten-
sing the curve along what seems to be its probable course, and measuring the
ordinates of the extended portions. It is not necessary that this extension
should coincide with what would be the true course of the curve in those parts.
The important point is, that the \( m \) terms thus added, taken together with the \( m+1 \) adjacent given terms, should follow a curve whose form is approximately algebraic and of a degree not higher than the third. As the adjusting process requires repeated multiplications by the coefficients \( l \), it will be well to prepare in advance a table showing the product of each of these coefficients by each of the nine digits. It will also answer every purpose, and save labor, if we adjust only alternate terms in the regular way, and then fill in the intermediate terms from these by the simple formula for "interpolation into the middle", which may be written

\[
u_0 = \lambda_0[9(u_1 + u_{-1}) - (u_3 + u_{-3})].
\]

(14)

Two additional tests of good adjustment have been discussed by the writer, one of which depends upon the fortuitous grouping of the \(+\) and \(-\) signs in the series of residual errors \( v \). \textit{(Interpolation and Adjustment of Series, p. 31.)} This is much the simplest test of any, its use requiring little labor or knowledge of mathematics. It was shown that in the case of a periodic series, that is, a series whose first and last terms are consecutive, the most probable number of isolated groups of \( n \) like signs, occurring from accidental causes, will be

\[
\frac{N}{2^{n+1}} \pm \frac{.6745}{2^{n+1}} \sqrt{\frac{(2^{n+1} - 1)N}{N}}.
\]

(15)

where the expression which follows the doubtful sign is the probable error. This will apply sufficiently well to the case of ordinary or non-periodic series, when the method of adjustment here discussed has been used, provided that we treat the first and last signs of \( v \) as consecutive, so that if they are alike, they belong to the same group. Taking \( n = 1 \), we find that the most probable number of isolated single signs, that is, signs unlike both the adjacent ones, is

\[
\frac{1}{4}N \pm \frac{.5545}{3N},
\]

(16)

and for \( n = 2 \), the most probable number of groups of two like signs is found to be

\[
\frac{1}{4}N \pm \frac{.5545}{7N},
\]

so that the most probable number of signs falling within groups of two is

\[
\frac{1}{4}N \pm \frac{.5545}{7N}.
\]

(17)

Adding (16) and (17) together, we have for the most probable number of signs falling within isolated groups of either one or two like signs

\[
\frac{1}{4}N \pm \frac{.5545}{10N},
\]

(18)

which may be written

\[
\frac{1}{4}N \pm .533\sqrt{N},
\]
and this is also the expression for the most probable number of signs falling in groups of more than two. Hence we have this practical rule, that if a series has been well adjusted, the whole number of signs of the residual $v$ which fall within groups of only one or two like signs each, will probably be about equal to the whole number which fall within groups of more than two, and the probable error of either number is

$$0.533v/N.$$ 

If the number of signs in groups of one or two should exceed $\frac{1}{2}N$ by more than this probable error, the inference will be that the series probably has not been smoothed out enough, whereas if they fall short of $\frac{1}{2}N$ by more than the probable error, we must presume that the series has been smoothed too much.

It was also shown in the treatise referred to, p. 33, that if in formula (15) the most probable number $N = 2^{n+1}$ is found to be less than $\frac{1}{2}$ for any assumed value of $n$, it will show that the odds are against the occurrence of any group of so many as $n$ like signs. If we take

$$\frac{1}{4} = \frac{N}{2^{n+1}}$$

it gives

$$n = 1 + \frac{\log N}{\log 2}, \ldots \ldots \ldots \ldots$$

that is to say

$$n = 1 + 3.32 \log N.$$ 

Hence we have the rule, that there will be a preponderance of chances against the occurrence, from accidental causes, of any group of a number of like signs greater than

$$1 + 3.32 \log N,$$

and if a larger group does occur, it will indicate a probability that the series has been smoothed out too much, or that its true law, in that vicinity, cannot be fairly represented by an algebraic curve of the third degree, for so many consecutive terms as are included by the adjustment formula.

The writer will take this opportunity to make a remark about the method of constructing equations of curves representing annual variations of temperature, from the monthly means taken as data, referred to at p. 314 of the *Smithsonian Report* of 1871. It was a means of suggesting to my mind a general method of interpolation, and has been referred to as the discovery of Professor Everett, who published it as such in the *Edinburgh New Philosophical Journal* for July 1861, and again in the *American Journal of Science and Arts* for January 1863. I have since learned that the method had...
been published some eleven years earlier, by M. Bravais, a French meteorologist. It may be found at p. 324 of Vol. II. of the *Meteorologie* which forms part of the series of the *Voyages de la Commission Scientifique du Nord*, edited for the French Government by M. Gaimard and others. The volume mentioned has no date, but one of the meteorological charts in the accompanying *Atlas de Physique* bears the date 1850.* The analytical process in question, therefore, ought to be designated after its first discoverer, as the method of Bravais, rather than as Everett's method. Of course this earlier origin of the method increases the probability that Schiaparelli was not unacquainted with it, as suggested by me in *Interpolation and Adjustment of Series*, p. 39. The same property has recently been published by a writer in the *Astronomische Nachrichten* for Nov. 24, 1873, and this is catalogued, apparently as a new discovery, in the *Jahrbuch über die Fortschritte der Mathematik* for that year, p. 123.

An inaccurate remark was made by me at p. 310 of the *Sm. Report* of 1871, in describing M. Tchebychef's mode of arranging data as intended for making "ordinary interpolations, not from groups, but from single terms or ordinates." From a brief allusion to the method in Bertrand's *Differential Calculus* I had supposed that it was something like what is known as the method of normal places. That this was an error, has been shown by a recent examination of Tchebychef's original memoir, which is entitled *Sur l'Interpolation dans le cas d'un grand nombre de données fournies par les Observations*. The method regards the series of observations as geometrically represented by ordinates, and the area of the polygon formed by joining their extremities is supposed to be divided by certain limiting ordinates into a number of areas, which are regarded as values of the area \( \int y \, dx \) for the required curve, between those limits.† These areas are taken as data for determining the values of the constants in the algebraic equation of the curve,

\[
y = A_0 + A_1 x + A_2 x^2 + \ldots + A_n x^n.
\]

The constant \( A_0 \) is determined from one set of areas, \( A_1 \) from another, and \( A_2 \) from still another, and so on, the limits of these areas being chosen with a view to securing the best values for the interpolated terms denoted by \( y \).

For further details, the reader should consult the original work, in the *Mémoires de l'Académie de Saint Pétersbourg*, 1859.

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*The observations were taken in the years 1838—40, and the results were published in a long series of *littératures*, both of the text and the charts, at different times for many years after the return of the expedition.

†In case observations are equidistant and very near together, each area is regarded as equal to the sum of all the observations falling within its limits, multiplied by the constant interval between two consecutive observations.
CURVE OF LOGARITHMIC SINES,

BY PROFESSOR L. G. BARBOUR, RICHMOND, KENTUCKY.

Let $ABD$ be a semicircle, $C$ its centre, radius = 1; and let fall from any number of points in the circumference sines, as $mn$, $op$, $qr$, $BC$, $st$, $uw$, &c. At $A$ and $D$ draw the tangents $AE$ and $DF$. Prolong the sines below the axis $AD$ until $nn' = \log mn$, $pp' = \log op$ &c. Through the extremities of these lines at $n'$, $p'$, $r'$, $C$, $t'$, $v'$ &c., draw a curve. It may be appropriately called the curve of logarithmic sines; $rr' = \log qr$; $pp' = \log op$; $nn' = \log mn$.

Taking $C$ as the origin of rectangular co-ordinates, and designating the lines $rr'$, $pp'$, $nn'$ &c., generally, by $y$, and the co-sines $Cr$, $Op$, $Ch$, &c., by $x$, we have

$$y = \log \sqrt{(1 - x^2)}$$

as the equation of the curve.

It is obvious that the curve has two symmetrical branches; that it is tangent to $AD$ at $C$; for $BC = 1$, \ldots, $\log BC = 0$; and that the tangent lines $AE$ and $DF$ are asymptotes. The logarithms are negative, as they should be for quantities less than 1. When the sine = 0, its logarithm = $-\infty$.

Length of Arc.—To get this we write $y = \log \sqrt{(1 - x^2)} = \frac{1}{2} \log (1 - x^2)$;

$$.\therefore dy = -\frac{x dx}{1 - x^2}. \quad \text{Putting arc} = s, \quad ds = dx \sqrt{\frac{1 - x^2 + x^4}{1 + 2x^2 + x^4}}.$$

$$s = \int_0^a dx \sqrt{\frac{1 - x^2 + x^4}{1 + 2x^2 + x^4}} = a + \frac{a^3}{6} + \frac{7a^5}{40} + \frac{17a^7}{112} + \frac{139a^9}{1152} + \frac{263a^{11}}{2816} + \&c.$$

Surface of Revolution.—Let $s =$ surface of revolution on $CG$ as an axis;

$$.\therefore ds = 2\pi xx ds.$$

$$s = \int_0^\pi dx \left[ x dx + \frac{1}{2} x^3 dx + \frac{7}{8} x^5 dx + \frac{17}{16} x^7 dx + \frac{139}{128} x^9 dx + \&c. \right]$$

$$= \pi \left[ \frac{x^2}{2} + \frac{x^4}{8} + \frac{7x^6}{48} + \frac{17x^8}{128} + \frac{139x^{10}}{1280} + \frac{263x^{12}}{3072} + \&c. \right]$$

Plane Areas.—We distinguish two of these; the interior, included within the arc $n'p'r'Ct'v'$; and the exterior, lying between the tangents $AE$, $DF$, the diameter $ACD$, and the curve. As both these areas are symmetrically divided by the axis $CG$, let us calculate first the interior area included between the axis $CG$ and the curve $Cr'p'n'$.
Designating this area by $I$, 
\[dI = xy = \int \frac{x^2 \log y}{1-x^2} = x \int \frac{d \log (1-x^2)}{1-x^2} = \int \frac{d \log (1-x^2)}{1-x^2} = \int \frac{1}{2(1-x)} \, dx - \int \frac{1}{2(1+x)} \, dx.\]

\[\therefore \quad I = \int \frac{-x^2 \, dx}{1-x^2} = x + \frac{1}{2} \log (1-x) - \frac{1}{2} \log (1+x).\]

Therefore the double interior area $= 2I = 2x + \log (1-x) - \log (1+x)$.

$x = 0$ gives area $= 0$. $\therefore \, s = 0$. $x = 1$ gives area $= 2 + \infty - \log 2 = \infty$.

Next, the exterior area, between $Orpn$, $nn'$ and $Or'p'n'$, designated by $E$.

\[dE = y \, dx = \frac{1}{2} \log (1-x^2) \, dx.\]

Integrating by parts,

\[\int \frac{1}{2} \log (1-x^2) \, dx = \frac{1}{2} \log (1-x^2) x - \int \frac{-x^2 \, dx}{1-x^2}.\]

$\therefore \quad E = xy - I$.

This is evidently correct, for the area of any rectangle as $Omn'H = xy$, and it is equal to the sum of $E + I$.

When $x = 0$, $E = 0$. When $x = 1$, $E = \log 2 - 1; \, 2E$, the double exterior area, $= 2 \log 2 - 2$. This is a remarkably simple expression, and furnishes another case, in addition to the "Witch" of Agnesi and a few other curves, of a finite area enclosed by lines some of which at least are infinitely long.

The proof however may be acceptable to some of my readers.

\[E = xy - I = xy - x - \frac{1}{2} \log (1-x) + \frac{1}{2} \log (1+x)\]

\[2E = x \log (1-x) - 2x - \log (1-x) + \log (1+x)\]

\[= x \log (1+x) + x \log (1-x) - 2x \quad \log (1+x) - \log (1-x)\]

\[= (1+x) \log (1+x) - (1-x) \log (1-x) - 2x.\]

Since when $x = 1, 1 - x = 0$, and $\log (1-x) = \infty$, the limit of their product is not apparent at the first glance; but it may be found by developing $\log (1-x)$ and then performing the multiplication by $(1-x)$. Thus:

\[\log (1-x) = -x - \frac{1}{2} x^2 - \frac{1}{3} x^3 - \frac{1}{4} x^4 - \cdots \]

\[-x \log (1-x) = \quad x^2 + \frac{1}{2} x^3 + \frac{1}{3} x^4 + \frac{1}{4} x^5 + \cdots \]

So long as $x$ is less than 1, the upper of these series is greater than the lower; but when $x = 1$, they are just equal and the term $(1-x) \log (1-x)$ $= 0$. \therefore for $x = 1$, $2E = (1+x) \log (1+x) - 2x = 2 \log 2 - 2$.

It will be observed that I have throughout used the Napierian logarithms.

In any system of logarithms,

\[2E = 2 \log 2 - 2M,\]

the logarithm of 2 being taken in that system. In the Napierian system, $M = 1$, and $\log 2 = .693147$; \therefore $E = .693147 - 1 = -.306853$.

**Logarithmoids.** — If the curve be revolved about $CG$ as an axis, it will describe a logarithmoid of revolution. $\, dx = xx^2 \, dy$; therefore
\[ v = \pi \int x^2 \, d \log (1 - x^2) = \pi \int \frac{-x^2 \, dx}{1 - x^2} = \pi (\frac{1}{2} x^2 + y) \]
in any system of logarithms. This may be called the interior volume.

An exterior volume is described by the exterior area \( \text{Onn}' \) revolving about \( CG \). Designating this by \( E.V., \) \( d E.V. = 2\pi yx \, dx = \pi \log (1 - x^2) \, dx. \)

Integrating by parts: \( \int u \, dv = uv - \int v \, du \). Let \( u = \log (1 - x^2), \) \( v = \frac{1}{2} x^2 \).

\[ du = \frac{-2x \, dx}{1 - x^2}, \]
\[ v = \frac{1}{2} x^2 \]
\[ \int \pi \log (1 - x^2) \, dx = \pi x^2 y - \frac{1}{2} x^2 y = -\pi [(1 - x^2) \frac{1}{2} \log (1 - x^2) + \frac{1}{2} x^2]. \]

When \( x = 1 \), it can be shown very much as in the former case that \((1 - x) \times \frac{1}{2} \log (1 - x^2) = 0; \) when \( x = 1, \) \( E.V. = \frac{1}{2} \pi. \)

Now the volume of the hemisphere described by revolving the quadrant \( \Delta BC \) about \( BC \) as an axis, \( = \frac{\pi}{2}; \) \( E.V. = \frac{1}{2} \) hemisphere.

**Singular Points.** — One has already been noted, the point of tangency of the curve with the axis of abscissas. It is characterized by \( \frac{dy}{dx} = 0. \) But \( \frac{dy}{dx} = \frac{-x}{1 - x^2}, \)

which is verified by \( x = 0. \)

Another is the point where a tangent to the curve makes an angle of 45° with the axes. Here \( \frac{dy}{dx} = \frac{-x}{1 - x^2} = -1. \) The sign of the 1 is negative in the 4th quadrant. \( x^2 + x = 1; \) \( 1 : x :: 1 + x : 1; \) \( 1 - x :: x :: x : 1. \)

That is, the abscissa of the point at which a tangent line, making an angle of 45° with either axis, touches the curve, is the greater segment of the radius when divided in extreme and mean ratio according to Euclid. Hence the abscissa = the chord of a decagon, twice the natural sine of 18°. 61804.

**Radius of Curvature.** — The equation is

\[ r = -\frac{\sqrt{(1 - x^2 + x^4)}}{1 - x^4}. \]

When \( x = 0 \) \( r = -1, \) \( \cdots \) the centre of the osculatory circle is on the axis of \( y, \) at the distance 1 below the centre of the original circle.

When \( x = 1, \) \( r = -\infty. \)

**The Anthoid.** — By revolving the curve \( C r' p' n' \) about the tangent line \( AE \) as an axis, we get a curved surface of a new pattern. The volume generated by the revolution of the surface \( n' p' r' C - A, \) may from its resemblance to flowers be called the Anthoid. — I leave these two last as problems for the readers of the Analyst.
SOLUTION OF PROBLEM 155.

BY CHAS. H. KUMMELL, U. S. LAKE SURVEY, DETROIT, MICH.

"To find the least distance between two places given by latitude and longitude, taking into account the polar compression."

This question admits two interpretations, viz.:—1. To find the rectilinear distance, that is the chord joining the two points; and 2. To find the shortest distance between the two points on the spheroid or that along the geodesic line.

**First Supposition.**—Let \( \varphi_1, \varphi_2 \) be the latitudes and \( \lambda_1, \lambda_2 \) the longitudes of the two given points respectively then we have their orthogonal coordinates \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) as follows:

\[
\begin{align*}
x_1 &= N_1 \cos \varphi_1 \cos \lambda_1, & x_2 &= N_2 \cos \varphi_2 \cos \lambda_2, \\
y_1 &= N_1 \cos \varphi_1 \sin \lambda_1, & y_2 &= N_2 \cos \varphi_2 \sin \lambda_2, \\
z_1 &= N_1 (1-\varepsilon^2) \sin \varphi_1, & z_2 &= N_2 (1-\varepsilon^2) \sin \varphi_2,
\end{align*}
\]

where \( N = \frac{a}{\sqrt{1-\varepsilon^2 \sin^2 \varphi}} \) = normal ending at polar axis, \( a \) = equatorial semiaxis and \( \varepsilon \) = eccentricity of meridian. We have then, denoting the straight line connecting the two points by \( s \),

\[
\begin{align*}
s &= \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2} \\
&= \sqrt{N_1^2 \cos^2 \varphi_1 - 2N_1N_2 \cos \varphi_1 \cos \varphi_2 \cos (\lambda_2 - \lambda_1) + N_2^2 \cos^2 \varphi_2} \\
&\quad + (1-\varepsilon^2)(N_2 \sin \varphi_2 - N_1 \sin \varphi_1)^2 \\
&= \sqrt{N_1^2 - 2N_1N_2[\cos \varphi_1 \cos \varphi_2 \cos (\lambda_2 - \lambda_1) + \sin \varphi_1 \sin \varphi_2]} \\
&\quad + N_2^2 - \varepsilon^2 (2-\varepsilon)(N_2 \sin \varphi_2 - N_1 \sin \varphi_1)^2 \}
\end{align*}
\]

(2)

If we draw through the center of the spheroid two lines parallel to the normals \( N_1, N_2 \) we have, denoting their included angle by \( \sigma \),

\[
\cos \sigma = \sin \varphi_1 \sin \varphi_2 + \cos \varphi_1 \cos \varphi_2 \cos (\lambda_2 - \lambda_1).
\]

(3)

We have then

\[
s = \sqrt{N_1^2 - 2N_1N_2 \cos \sigma + N_2^2 - \varepsilon^2 (2-\varepsilon)(N_2 \sin \varphi_2 - N_1 \sin \varphi_1)^2}.\]

(4)

Join the ends of the normals by a straight line which may be denoted by \( s' \), then we have from the rectilinear triangle thus formed

\[
s' = \sqrt{N_1^2 - 2N_1N_2 \cos \sigma + N_2^2},
\]

(5)

hence

\[
s = \sqrt{(s')^2 - \varepsilon^2 (2-\varepsilon)(N_2 \sin \varphi_2 - N_1 \sin \varphi_1)^2}.
\]

(6)

The geometrical signification of the second term under the radical, which is always a small quantity and may be neglected for two places of nearly the same latitude, is the following:
Let \( ds' = N_2 \sin \varphi_2 - N_1 \sin \varphi_1 \) = projection of \( s' \) on polar axis, 
\[ ds = z_2 - z_1 = (1 - e^2)(N_2 \sin \varphi_2 - N_1 \sin \varphi_1) \] = projection of \( s \) on polar axis, then 
\[ ds'^2 - ds^2 = e^2(2 - e^2)(N_2 \sin \varphi_2 - N_1 \sin \varphi_1)^2 \] and 
\[ s = \sqrt{[(s')^2 - ds'^2 + ds^2]}, \] 
(7) 
\[
\text{hence} \\
\[ s^2 - ds^2 = s^2 - ds'^2, \] 
(8) 
that is, the projections of \( s \) and \( s' \) on the equator are equal.

Second Supposition—Any surface of revolution may be represented by the equation 
\[ z = f(\rho), \] 
(1) 
where \( z \) = distance of any point from some plane of reference at right angles to the axis of rotation, which may be called the equator, and \( \rho \) = radius of parallel. In this form the equation of the surface is independent of the third coordinate, the longitude \( \lambda \) = inclination of any meridian to the first meridian. In this system of coordinates we have for the element of arc of any curve in space 
\[ ds = \sqrt{[d\rho^2 + \rho^2 d\lambda^2 + dz^2]}, \] 
(2) 
and the arc of any curve between two fixed points \((\rho_1, \lambda_1, z_1), (\rho_2, \lambda_2, z_2)\) 
\[ s = \int_1^2 \sqrt{[d\rho^2 + \rho^2 d\lambda^2 + dz^2]} \] 
(3) 
If this arc is to be on the surface \( z = f(\rho) \) there will be only two independent variables, viz., \( \rho \) and \( \lambda \) in (3); and if we introduce the condition that \( s \) shall be the shortest arc on the surface between the given points we must have \( \frac{\partial s}{\partial \lambda} = 0 \), or else \( \frac{\partial s}{\partial \rho} = 0 \).

But 
\[ \frac{\partial s}{\partial \lambda} = \frac{1}{\rho \frac{d\rho}{ds}} \int_1^2 \rho^2 d\lambda \frac{d\lambda}{ds} = \left[ \frac{\rho^2 d\lambda}{ds} \right]_1^2 \frac{1}{\rho \frac{d\rho}{ds}} \int_1^2 \left( d \cdot \frac{\rho^2 d\lambda}{ds} \right) d\lambda = 0. \]

The first term vanishes at the limits since the curve is to pass through fixed points, consequently the other term must be 0 also, and therefore 
\[ d \cdot \frac{\rho^2 d\lambda}{ds} = 0, \text{ or, integrating } \rho^2 d\lambda = c d s, \] 
(4) 
and integrating again
\[ \int_1^2 \rho^2 d\lambda = cs. \] 
(5) 
This equation shows that the arc of the curve is proportional to the sector of its horizontal projection.

If \( A \) is the angle which the geodesic line makes with the meridian or its azimuth, we have evidently
\[ \sin A = \rho \frac{d\lambda}{ds} \] 
(6) Combining this equation with (4) we have
\[ \rho \sin A = c. \] 
(7)
The constant $c$ is therefore the radius of parallel where the geodesic line meets a meridian at right angles. Combining (4) with (2) we obtain
\[ d\lambda = \frac{c}{\rho} \sqrt{\left(\frac{dp^2 + dz^2}{\rho^2 - \sigma^2}\right)}, \]  
which is the polar differential equation of the horizontal projection of the geodesic line on any surface of revolution; for it is an equation between the longitude $\lambda$, the radius of parallel $\rho$ and an undetermined constant $c$. To determine this constant we have the condition
\[ \lambda_2 - \lambda_1 = c \int_{1}^{21} \sqrt{\left(\frac{dp^2 + dz^2}{\rho^2 - \sigma^2}\right)}. \]  
The length of the arc of the geodesic is then
\[ s = \int_{1}^{2} \rho \sqrt{\left(\frac{dp^2 + dz^2}{\rho^2 - \sigma^2}\right)}. \]  
For the spheroid we have
\[ z = \sqrt{\left[(1 - e^2)(\alpha^2 - \rho^2)\right]}; \]
\[ \lambda_2 - \lambda_1 = c \int_{1}^{2} \frac{dp}{\rho} \sqrt{\left[\frac{e^2 \rho^2}{(\alpha^2 - \rho^2)(\rho^2 - \sigma^2)}\right]}, \]
\[ s = \int_{1}^{2} \rho \sqrt{\left[\frac{e^2 \rho^2}{(\alpha^2 - \rho^2)(\rho^2 - \sigma^2)}\right]} \].

The latitudes and longitudes of the two points being the data, we have for the limits of these integrals
\[ \rho_1 = a (\cos \varphi_1) + d \varphi_1, \quad \rho_2 = a (\cos \varphi_2) + d \varphi_2, \]
if \( d \varphi = \sqrt{(1 - e^2)\sin^2 \varphi} \).

Placing $\rho = \sqrt{\left[\alpha^2 + (\alpha^2 - \sigma^2)\sin^2 \psi\right]}$ we have
\[ \lambda_2 - \lambda_1 = c \int_{1}^{2} \frac{dp}{\alpha^2 - \sigma^2} \sqrt{\left[\frac{\alpha^2 - \sigma^2}{\alpha^2 - \sigma^2}(\alpha^2 - \sigma^2)\sin^2 \psi\right]} = \int_{1}^{2} \frac{F_2(\psi, c \sqrt{\left(\frac{\alpha^2 - \sigma^2}{\alpha^2 - \sigma^2}\right)})}{\sqrt{(\alpha^2 - \sigma^2)}} \]
\[ s = \int_{1}^{2} \frac{dp}{\alpha^2 - \sigma^2} \sqrt{\left[(\alpha^2 - \sigma^2)\sin^2 \psi\right]} = \sqrt{(\alpha^2 - \sigma^2)}E_2(\psi, \psi, \frac{\alpha^2 - \sigma^2}{\sqrt{(\alpha^2 - \sigma^2)}}) \]
\[ \ldots \]

These elliptics give the complete solution of the problem in theory. The great difficulty consists in the determination of the constant $c$. If the azimuth of the geodesics were identical with the azimuth as measured by a theodolite and which is the inclination of the vertical circle, passing through the other point, to the meridian;* then equation (7) would give the constant $c$.  

*This field azimuth is given in terms of the latitudes and longitudes of the two points by the formula: \( \cot A_1 = \cot \alpha_1 - \frac{\rho_2 \cos \phi_1}{\sin(\lambda_2 - \lambda_1) \cos \phi_1} \cdot \sin \phi_2 - \frac{\sin \phi_2}{\Delta \phi_2} \cdot \frac{\sin \phi_1}{\Delta \phi_1} \), where \( \cot \alpha_1 = [\tan \phi_1 \times \cos(\phi_1 - \sin(\lambda_2 - \lambda_1) + \sin(\lambda_2 - \lambda_1)]. \) It may differ from the azimuth of the geodesic line by any amount between 0° and 90°. For instance, if both points are on the equator and nearly 180° apart the geodesic line passes near the pole, its azimuth is therefore nearly 180° while the field azimuth (of course impotent) would follow the plane of the equator and is=270°.
without any difficulty and equation (9) would be superfluous. If the two points are not too distant a value for \( c \) from (7) may be used as a first approximation and its correction determined by (9). The constant \( c \) being thus found, the length of the arc is found by the evaluation of an elliptic of the second species.

\[ \text{\textit{Note.---As it is contended that in the published answer to Prof. Hall's Query (see p. 94) the series which represents the value of } u \text{ converges so slowly that the method is inconvenient, another answer is here submitted as given by Chas. H. Kummell.}} \]

To find the most convenient way of computing the numerical value of the definite integral

\[ I = \int_{0}^{\pi} \frac{d\varphi}{(\sin \varphi)}. \]  

(1)

We have

\[ \int_{0}^{\pi} (\sin \varphi)^{2m-1} (\cos \varphi)^{2n-1} = \frac{1}{2} B(m, n) = \frac{\Gamma(m) \Gamma(n)}{2 \Gamma(m+n)}. \]  

(2)

Placing \( m = \frac{1}{4} \) and \( n = \frac{1}{4} \) we have

\[ I = \int_{0}^{\pi} \frac{d\varphi}{(\sin \varphi)} = \frac{\Gamma(\frac{1}{4}) \Gamma(\frac{1}{4})}{2 \Gamma(\frac{1}{2})}. \]  

(3)

But

\[ \Gamma(\frac{1}{4}) = \sqrt{\pi} \]

\[ \Gamma(\frac{1}{4}) = \pi^{\frac{1}{2}} I(\frac{1}{4}) \]

by the theorem: \( \Gamma(n) \Gamma(1-n) = \frac{\pi}{\sin \pi n} \)

\[ \Gamma(\frac{1}{4}) = \frac{1}{2} \Gamma(\frac{1}{4}) \]

therefore

\[ I = \frac{(2\pi)^{\frac{1}{4}}}{\Gamma(\frac{1}{4})}. \]  

(4)

The circumference of the lemniscate of Bernoulli may be expressed in terms of \( \Gamma(\frac{1}{4}) \) as follows:

The polar equation of the lemniscate is \( r^2 = a^2 \cos 2\theta \). Denoting the circumference by \( P \) we have

\[ \frac{1}{2} P = a \int_{0}^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\cos 2\theta}}. \]  

(5)

Placing \( \theta = \frac{1}{2} \pi - \frac{1}{2} \varphi \) we have

\[ \frac{1}{2} P = \frac{a}{2} \int_{0}^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{\sin \varphi}} = \frac{1}{4} a B(\frac{1}{4}, \frac{1}{4}) = \frac{a}{4} \frac{\Gamma(\frac{1}{4}) \Gamma(\frac{1}{4})}{\Gamma(\frac{1}{2})} \text{ [by (2)]}, \]

or

\[ P = a \frac{\Gamma(\frac{1}{4})^{2}}{\sqrt{2\pi}}. \]  

(6)

Combining this with (4) we obtain

\[ I = \frac{2a\pi}{P}. \]  

(7)
The proposed integral is therefore found by dividing the circumference of the lemniscate into the circumscribed circle.

To determine now $P$ place in (5) $\sin \theta = \sqrt{\frac{1}{2} \times \sin \phi}$, then

$$\frac{1}{4}P = a \int_{\frac{\pi}{2}}^{\phi_0} \frac{d\phi}{\sqrt{(a^2_0 \cos^2 \phi + b^2_0 \sin^2 \phi)}}.$$  

(8)

Take the more general form

$$E_o = \int_{0}^{\psi_o} \frac{d\psi}{\sqrt{(a^2 \cos^2 \psi + b^2 \sin^2 \psi)}}.$$  

(9)

which reduces to (7) by placing $a_o = \sqrt{2}$, $b_o = 1$ and $\psi_o = \frac{1}{2} \pi$.

By Landen's substitution, let

$$\tan(\phi_0 - \phi) = (b_0 + a_0) \tan \phi_0, \quad a_1 = \frac{1}{2}(a_0 + b_0); \quad b_1 = \sqrt{(a_0 b_0)};$$  

(10)

then

$$E_o = \frac{1}{2} \int_{0}^{\psi_1} \frac{d\phi}{\sqrt{(a_1^2 \cos^2 \phi + b_1^2 \sin^2 \phi)}} = \frac{1}{2} E_1.$$  

(12)

Similarly let

$$\tan(\phi_2 - \phi_1) = (b_1 + a_1) \tan \phi_1, \quad a_2 = \frac{1}{2}(a_1 + b_1); \quad b_2 = \sqrt{(a_1 b_1)};$$  

(10')

then

$$E_1 = \frac{1}{2} \int_{0}^{\psi_2} \frac{d\phi}{\sqrt{(a_2^2 \cos^2 \phi + b_2^2 \sin^2 \phi)}} = \frac{1}{2} E_2.$$  

(12')

Continuing this process until $a_n = b_n$ we have finally

$$E_n = \frac{\psi_n}{b_n}; \quad \cdots \quad E_0 = \frac{\psi_0}{2b_n}.$$  

(13)

If $\psi_0 = \frac{1}{2} \pi$ then $\phi_1 = \pi$, $\phi_2 = 2\pi$ \ldots $\phi_n = 2^{n-1} \pi$; therefore

$$\frac{1}{4}P = \frac{\alpha \pi}{2b_n}.$$  

(14)

Substituting this into (7) we have

$I = b_n =$ arithmetic-geometric mean of $\sqrt{2}$ and 1.

The very convenient computation may be arranged as follows:

\[
\begin{align*}
 a_0 &= 1.4142135624 & \log b_0 &= 0.0000000.000 \\
 b_0 &= 1.0000000000 & \log a_0 &= 0.1505149.879 \\
 a_1 &= 1.2071067812 & \log b_1 &= 0.0752574.989 \\
 b_1 &= 1.1892071150 & \log a_1 &= 0.0817456.897 \\
 a_2 &= 1.1981569481 & \log b_2 &= 0.0785015.943 \\
 b_2 &= 1.1981235214 & \log a_2 &= 0.0785137.106 \\
 a_3 &= 1.1981402347 & \log b_3 &= 0.0785076.525 \\
 b_3 &= 1.19811402347 & \log a_3 &= 0.0785076.525 \\
 \end{align*}
\]

\[
\begin{align*}
 \log b_4 &= 0.0785076.525 = \log b_n. \\
 \cdots \int_{0}^{\frac{\pi}{2}} \frac{d\phi}{\sin \phi} &= 1.1981402347.
\end{align*}
\]
DEMONSTRATION OF PROP. XXV. OF THE FIRST BOOK OF EUCLID.

BY W. E. HEAL, WHEELING, INDIANA.

The twenty-fifth proposition of the first book of Euclid, which that author (and I believe, every succeeding geometer) demonstrates indirectly, that is by the reductio ad absurdum, may be demonstrated directly as follows:

Theorem:—If two triangles have two sides of the one equal to two sides of the other, each to each, but the bases unequal, the angle contained by the sides of that which has the greater base will be greater than the angle contained by the sides of the other.

In the triangles $ABC, DEF$ let $AB = DE, AC = DF$ and $BC$ be greater than $EF$, then shall the angle $BAC$ be greater than the angle $EDF$.

Of the sides $AB, AC$ let $AB$ be the one which is not greater than the other, and from the base $BC$ cut off a part $BG = EF$ and adjacent to $AB$; on $BG$ construct the triangle $BHG$ so that $BH = DE$ (or $AB$) and $HG = DF$ (or $AC$); join $AH$ and $HC$. Because $BH = BA$, the angle $BHA$ = the angle $BAH$. And since the side $BH$ is not greater than $HG$ the angle $HGB$ is not greater than $HBG$; but the angle $HGC$ is greater than $HGB$; therefore the angle $HGC$ is greater than $HGB$; but $HGB$ is greater than $HCG$; much more, then, is the angle $HGC$ greater than $HCG$. And because the angle $HGC$ is greater than $HCG$ the side $HC$ is greater than the side $HG$ or its equal $AC$. Then, in the triangle $HCA$, the angle $HAC$ is greater than the angle $AHC$ because the side $HC$ is greater than $AC$. To these unequal angles add the equals $BHA$ and $BAH$ and there results angle $BAC$ greater than angle $BHC$; but $BHG = EDF$; therefore the angle $BAC$ is greater than the angle $EDF$; which was to be proved.

QUERY, BY THE EDITOR. — As we have been requested to furnish a demonstration to the following proposition, the subjoined demonstration is submitted; and we present to our readers the query: By whom was the proposition originally announced? — Also, give the author's demonstration.
Proposition.—"$D + d = a \cot A + b \cot B + c \cot C$, in which $d$ and $D$ are the diameters of the circles inscribed in, and circumscribed about, a triangle whose sides are $a$, $b$, $c$, and opposite angles $A$, $B$, $C$.”

Demonstration.—Let $ABC$ represent the triangle and $O$, the center of the circumscribing circle, and let $D$, $E$, $F$, represent the middle points of the sides $a$, $b$, $c$.

Because the $\angle BOC$ at the center is double the $\angle A$ at the circumference, $\angle BOD = \angle A$.

\[ DO = x = \frac{1}{2}a \cot A, \text{ and similarly} \]
\[ EO = y = \frac{1}{2}b \cot B, \]
\[ FO = z = \frac{1}{2}c \cot C. \]

Join $DE$, $DF$ and $EF$; then is $EF = \frac{1}{2}a$, $DF = \frac{1}{2}b$ and $DE = \frac{1}{2}c$.

Because the quadrilaterals $DCEO$, $EAFO$ and $FBDO$ have, each, the sum of their opposite angles $= 180^\circ$ they are inscribable in a circle.

Therefore (Eucl. D. VI.) we have
\[ \frac{1}{2}aR = \frac{1}{2}xy + \frac{1}{2}bz, \]
\[ \frac{1}{2}bR = \frac{1}{2}cx + \frac{1}{2}az, \]
\[ \frac{1}{2}cR = \frac{1}{2}ay + \frac{1}{2}bx. \]

\[ R(a + b + c) = (b + c)x + (a + c)y + (a + b)z, \quad (1) \]

where $R$ is the radius of the circumscribing circle.

Let $r$ represent the radius of the inscribed circle, then, obviously,
\[ r(a + b + c) = ax + by + cz. \quad (2) \]

Adding (1) and (2) we get
\[ (a + b + c)(R + r) = (a + b + c)x + (a + b + c)y + (a + b + c)z, \]
or
\[ R + r = x + y + z = \frac{1}{2}a \cot A + \frac{1}{2}b \cot B + \frac{1}{2}c \cot C. \]

\[ D + d = a \cot A + b \cot B + c \cot C. \]

SOLUTIONS OF PROBLEMS IN NUMBER THREE.

Solutions of problems in No. 3 have been received as follows:

From Marcus Baker, 162 and 164; Prof. L. G. Barbour, 162; Prof. W. P. Casey, 166; George Eastwood, 164 and 166; Prof. H. T. Eddy, 165 & 168; Henry Gunder, 162; W. E. Heal, 162; Chas. H. Kummell, 162, 168 and 169; Prof. O. Pratt, Sen., 163; J. W. P. Reid, 162; L. Regan, 162; E. B. Seitz, 162 and 164; Prof. C. M. Woodward, 167; A. W. Whitaker, 162; F. A. Walker, 162.

162. "Given
\[ x^3 + xy + y^2 = 37, \quad (1) \]
\[ x^3 + xz + z^2 = 49, \quad (2) \]
\[ y^3 + yz + z^2 = 61, \quad (3) \]

to find $x$, $y$ and $z$ by quadratics".
SOLUTION BY W. E. HEAL, WHEELING, INDIANA.

From (1), (2) and (3) we get, by multiplication,

\[ x^2 - y^2 = 37(x - y), \]  
\[ x^2 - z^2 = 49(x - z), \]  
\[ y^2 - z^2 = 61(y - z). \]  

\[ \therefore 37(x - y) = 49(x - z) - 61(y - z). \]

Reducing,

\[ 2y = x + z. \]

Put \[ x = y - v \]  \(9)\), and \[ z = y + v. \]  \(10)\)

Substituting in (3) and (2) we have

\[ 3y^2 + 3yv + v^2 = 61, \]  
\[ 3y^2 + v^2 = 49. \]  

\[ \frac{3yv}{(12)} = 12, \text{ or, } yv = 4. \]  \(13)\)

From (12) and (13)

\[ \sqrt{[(12) - 12(13)^2]} = \pm (3y^2 - v^2) = 47; \]  
\[ \frac{4}{12}(12) + \frac{4}{14}] = y = \pm 4 \text{ or } v = \pm 4\sqrt{3}, \]  
\[ \sqrt{[\frac{4}{12} - \frac{4}{14}] = v = \pm 1 \text{ or } y = \pm \frac{1}{3}\sqrt{3}. \]  

\[ \therefore x = \pm 3 \text{ or } \pm \frac{1}{3}\sqrt{3}, \text{ and } z = \pm 5 \text{ or } \pm \frac{1}{3}\sqrt{3}. \]

163 “Resolve the first member of the general Cubic Equation, \(x^3 + px^2 + qx = - r\), into three factors, such that, when their signs are changed, their sum shall equal \(p\); when their signs are unchanged the sum of their products, taken two and two, shall equal \(q\); and when their signs are unchanged, their continued product shall \(- r\). Or, in other words, find the forms of the three roots in terms of \(x\) and the coefficients.”

SOLUTION BY PROF. OBSON PRATT, SEN., SALT LAKE CITY, UTAH.

The following three factors fulfil all the conditions required in the problem, and, therefore, must be the true forms of the three roots:

\[ x; \]
\[-\frac{1}{2}p - \frac{1}{2}x + \frac{1}{2}\sqrt{[p^2 - 4q - 2px - 3x^2];} \]
\[-\frac{1}{2}p - \frac{1}{2}x - \frac{1}{2}\sqrt{[p^2 - 4q - 2px - 3x^2].} \]

164. “From any two points to draw two lines which shall meet in the circumference of a given circle, and make equal angles with the tangent at the point of intersection.”

SOLUTION BY GEORGE EASTWOOD, SAXONVILLE, MASS.

Analysis: Suppose the problem solved; \(A\) and \(B\) the given points, \(O\) in \(AB\) the centre of the given circle, \(P\) the required point in its circumference.
and \( AP, BP \) lines drawn making equal angles \( APC, BPD \) with the indefinite tangent \( CD \).

Join \( OP \); then \( OP \) is perpendicular to \( CD \) (Eucl. III. 18), the angles \( OPC, OPD \) are right angles, the angles \( APC, BPD \) are equal (by hypothesis), therefore \( \angle APO = \angle BPO \). Hence \( PO \) divides \( AB \) in \( O \) so that

\[ \frac{AO}{OB} :: \frac{AP}{BP}. \] (Eucl. VI. 3.)

Now \( AO \) and \( OB \) are given lines, therefore \( AP : PB \) is a given ratio. Let it be as \( 1 : n \), so that \( n \cdot AP = BP \). But (Eucl. VI. B.)

\[ AP \cdot PB = AO \cdot OB + OP^2, \]

\[ = AO(AO + BO) = AO \cdot AB, \] a given magnitude, therefore \( AP^2 = AO \times AB + n \), a given rectangle, therefore \( AP \) is known. Hence the following

Construction. On \( AB \), the line joining the given points, take \( AO \) equal to the radius of the given circle. With centre \( A \) and a radius equal to a mean proportional between \( AO \) and the \( n \)th part of \( AB \), describe an arc intersecting the given circumference in \( P \). Join \( OP \) and erect the indefinite perpendicular \( CPD \). Join \( AP, BP \), then the angles they respectively make with \( CPD \) will be equal to each other.

Scholium. Let \( AP \) be produced (downward) to \( A' \), \( BP \) to \( B' \), and \( OP \) to \( O' \) in \( A'B' \). Then if \( A' \) and \( B' \) are taken for the given points, it may be easily shown that the lines \( A'P \) and \( B'P \) will make equal angles with the tangent \( CPD \).

[Because, in the foregoing solution, one of the given points must lie in the circumference of the given circle, therefore, if by "a given circle" is meant a circle given both in magnitude and position, this solution is but a particular case of a more general solution involving an equation of the 4th degree. This is the view taken by E. B. Seitz and Marcus Baker, each of whom has reduced the solution to an algebraic equation of the 4th degree. Mr. Baker remarks, "This is the same as the celebrated problem of Alhazen. It was the prize question in the Ladies' Diary for 1727. Solutions of it have been given by Slusius, Huyghens, Dr. Robert Smith, Robert Simson, Robins, Hutton, Wales, &c."—Ed.]

165. "Of the system of dynamical equations,

\[ \frac{d^2x}{dr^2} + \frac{mx}{r^3} = 0, \quad \frac{d^2y}{dr^2} + \frac{my}{r^3} = 0, \quad \frac{d^2z}{dr^2} + \frac{mz}{r^3} = 0, \]

where \( r = \sqrt{(x^2 + y^2 + z^2)} \), seven first integrals are obtained of which it is subsequently found that five only are independent. How many final integrals can hence be deduced without proceeding to another integration."
SOLUTION BY PROF. H. T. EDDY, UNIVERSITY OF CINCINNATI.

The differential equations of the relative motion of two particles, having a combined mass of \( m \), due to their mutual gravitation, are:

\[
\frac{d^2 x}{dt^2} + \frac{mx}{r^3} = 0, \quad \frac{d^2 y}{dt^2} + \frac{my}{r^3} = 0, \quad \frac{d^2 z}{dt^2} + \frac{mz}{r^3} = 0,
\]
in which \( r^2 = x^2 + y^2 + z^2 \). Multiply severally by \( dx \), \( dy \), \( dz \), and integrate

\[
\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2} = \frac{ds^2}{dt^2} = \frac{2m}{r} + a.
\]

Again, multiply severally by either \( x, y \) or \( z \) and substitute, i.e., eliminate,

\[
z \frac{dy}{dt} - y \frac{dz}{dt} = 0, \quad \&c.
\]

\[
\frac{dz}{dt} - y \frac{dz}{dt} = b_1 \quad (2), \quad \frac{dx}{dt} - z \frac{dz}{dt} = b_2 \quad (3), \quad \frac{dx}{dt} - x \frac{dy}{dt} = b_3. \quad (4)
\]

Once more, multiply the given equations severally by either (2), (3) or (4) and substitute,

\[
b_2 \frac{dz}{dt} - b_3 \frac{dy}{dt} = - \left( \frac{z}{dt} - \frac{dx}{dt} \right) mx + \left( \frac{y}{dt} - \frac{dx}{dt} \right) my.
\]

\[
b_2 \frac{dz}{dt} - b_3 \frac{dy}{dt} = m \int \left[ \frac{x^2 + y^2 + z^2}{r^3} dx - x(adx + ydy + zdz) \right]
\]

\[
b_2 \frac{dz}{dt} - b_3 \frac{dy}{dt} = m \int \frac{rdx - xdr}{r} = m \int \frac{dx}{r}.
\]

\[
b_2 \frac{dz}{dt} - b_3 \frac{dy}{dt} = \frac{m}{r} x + c_1,
\]

\[
b_2 \frac{dx}{dt} - b_1 \frac{dz}{dt} = \frac{m}{r} y + c_2,
\]

\[
b_1 \frac{dy}{dt} - b_2 \frac{dx}{dt} = \frac{m}{r} z + c_3.
\]

Equations (1) to (7) inclusive are the seven first integrals.

Now, multiply severally (2), (3), (4), by \( x, y, z \) and add,

\[
\frac{b_1 x + b_2 y + b_3 z = 0.}
\]

Also, multiply (5), (6), (7) severally by \( x, y, z \), add and reduce by (2), (3), (4),

\[
\frac{mr + c_1 x + c_2 y + c_3 z = b_1^2 + b_2^2 + b_3^2.}
\]

Equations (8) and (9) represent a plane and quadric respectively whose intersection is the orbit: there are therefore no other independent relations between \( x, y, z \). There must be however three more equations expressing \( x, y, z \) in terms of \( t \): these cannot however be determined by any elimination between the seven first integrals, because \( t \) only appears in them as a differential. Only two of these five independent primitives can therefore be obtained without further integration.
166. "If through a point $O$, within a given triangle $ABC$, three lines be drawn respectively parallel to the sides of the triangle; viz., $GE$ parallel to $BC$, $FH$ to $AB$ and $DT$ to $AC$, and there is given, $GO \times OE + FO \times OH + DO \times OT$; to find the locus of $O$ by plane geometry."

**SOLUTION BY PROF. W. P. CASEY, SAN FRANCISCO, CAL.**

**Construction.** Let $X$ be the centre of the circle circumscribing the triangle $ABC$. Join $AO$ and produce it to $K$, draw $TR$ parallel to $OK$, and $TI$ parallel to $CK$. Join $IF$; and make the angle $ADY = \angle OIF$.

**Demonstration.** It is evident that a circle will circumscribe the quadrilateral $OFIT$, and that the triangles $GOA$ and $TRC$ are similar; also, the triangles $ODY$ and $OIT$ are similar. Hence we have $GO \times TC = AO \times TR$, or $GO \times OE = AO \times IK$ (1); and from the similar triangles $ADY$ and $OIF$ we have $AD \times OF = AY \times OI$, and therefore $HO \times OF = AY \times OI$ (2); and from the similar triangles $ODY$ and $OIT$ we have $DO \times OT = OY \times OI$ (3). By adding equations (1), (2) and (3) we get $GO \times OE + HO \times OF + DO \times OT = AO \times IK + AY \times OI + OY \times OI = AO \times IK + AO \times OI = AO \times OK = (XK)^2 - (XO)^2 = a$ given quantity.

[Solutions of the remaining problems in No. 3 will be published in No. 5, as follows: 167, by Prof. C. M. Woodward; 168, by Prof. H. T. Eddy; 169, by Chas. H. Kummell.—No answers to Queries in No. 3 have been rec’d.]

**PROBLEMS.**

160. (SELECTED) BY G. W. HILL. — Given the lengths of the eight edges of a quadrangular pyramid to find its altitude.

161. BY O. H. MERRILL, SOUTH RUTLAND, N. Y. — A pail in the form of a frustrum of a cone—radius of upper base $2r$, of lower base $r$, and height $h$—is inclined so that it water be poured in to it, the water will just reach the lower edge of the upper base, and the upper edge of the lower base. How many gallons of water in the pail?

162. BY ARTEMAS MARTIN, ERIE, PA. — Find the least integral values of $x$ and $y$ that will satisfy the equation $x^2 - 9817y^2 = 1$. 
163. By E. B. Seitz, Greenville, Ohio.—A given semicircle is divided into two quadrants, and a point is taken at random in each quadrant; find the chance that the distance between them is less than the radius of the semicircle.

164. By Chas. H. Kummell, Detroit, Mich.—Give the most convenient method to compute $I^{(\frac{1}{4}n)}$, $n$ being an integer.

Query. By Marcus Bakek, Washington, D. C.—In the treatise on the elements of Geometry by Rouche and Comberousse, some miscellaneous problems in solid geometry are proposed, among which is the following:—Draw a sphere which shall cut five other spheres under equal angles.

Can any reader of the Analyst refer me to a solution, or if not, give one.

NOTICES OF PUBLICATIONS RECEIVED.

On Eight Meteoric Fireballs seen in the United States from July, 1876, to February, 1877. By Daniel Kirkwood, LL. D., Professor of Mathematics in Indiana University, Bloomington Indiana. (Read before the American Philosophical Society, March 16, 1877.)

On the Relative Ages of the Sun and certain of the Fixed Stars. By Professor Daniel Kirkwood, of Indiana University. (Read before the American Philosophical Society, April 6, 1877.)

Trilinear Coordinates; being No. II. of Mathematical Tracts Relating to the Modern Higher Mathematics. By Rev. W. J. Wright, Ph. D., Member of the London Philosophical Society. London: C. F. Hodgson and son, Gough square, Fleet street. 1877. This is an 8vo pamphlet of 77 pages, and is devoted, mainly, as its title imports, to illustrations of the various applications of the Trilinear System of Coordinates. —It would be difficult to present, in the same space, a clearer discussion of the subjects treated; and, besides, the publishers have contributed to the value of the book by the use of good paper and a faultless typography.


On the Part of the Motion of the Lunar Perigee which is a Function of the mean motions of the Sun and Moon. By G. W. Hill, Ph. D. 4to. 28 pp. Cambridge. 1877.

The rate of motion of the Lunar Perigee can be expanded in a series of ascending powers and products of the squares of the following quantities;—the lunar and solar excentricities, the lunar inclination and the ratio of the solar to the lunar parallax. The object of the author is to compute, with very great exactitude, (15 decimals are used), the first term of this series. By means of an integral, discovered by Jacobi, and of a particular solution, apparently due to the author, the question is reduced from depending on a linear differential equation of the 4th order to one of the 2nd. The treatment of the latter in § III of the memoir is extremely novel and well adapted to the end in view. But the method pursued demands the previous determination of the lunar inequalities having the argument of the variation. For this the author refers us to a yet unpublished memoir to which the present paper is properly the sequel. On this account much of the paper will, with difficulty, be intelligible, except to those having considerable familiarity with the lunar theory. For this reason, the author should not delay to give us the remaining portion of his researches.
THE ANALYST.

Vol. IV. September, 1877. No. 5.

SHORT METHOD OF ELLIPTIC FUNCTIONS.

BY LEVI W. MEECH, A. M., HARTFORD, CONN.

The system of Elliptic Functions is closely connected with analytical geometry and trigonometry, and may be said to rank next in importance. The earlier advances of Fagnani, Euler, Landen and Lagrange were expanded in a systematic treatise by Legendre, whose well directed labor of forty years is also allied with the brilliant discoveries of Abel and Jacobi, and many others. Their researches have greatly facilitated the solution of numerous problems in Mechanics, Astronomy, Geometry, and the theory of Heat and Light.

The present compendium has been arranged as free from complexity as possible. Especially in the seventh Section, the unexpected discovery of a new scale of amplitudes adapted equally to successive or to simultaneous quadrature, happily attains the object of the theorems of Jacobi and Ivory, without the previous long and intricate demonstration. And the common origin of all known scales of moduli is traced to a simple integral in the next Section.

SECTION I. Change from Algebraic to Elliptic Functions.
  II. Connection with the Spherical Triangle.
  III. Increasing Scales of Moduli.
  IV. Decreasing Scales of Moduli.
  V. Integration by series. Table of Quadrants.
  VI. Inverse Method of Integration. Theta Functions.
  VII. Successive and Simultaneous Quadratures.
  VIII. Quadrantal Quadrature.
  IX. Elliptic Functions of the Second and Third Order.
DEFINITIONS. Elliptic Functions, or Transcendents, were classed by Legendre into three species. And the complete periphery of each is a closed curve having four similar quadrants. The Function of the first species is denoted by \( F \). The second, which is an arc of the common ellipse, is denoted by \( E \); and the third by \( \Pi \). Thus,

\[
F = F(e, \theta) = \int_0^\theta \frac{d\theta}{\sqrt{1 - e^2 \sin^2 \theta}}.
\]

\[
E = E(e, \theta) = \int_0^\theta \sqrt{1 - e^2 \sin^2 \theta} d\theta.
\]

\[
\Pi = \Pi(e, n, \theta) = \int_0^\theta \frac{d\theta}{(1 + n \sin^2 \theta)\sqrt{1 - e^2 \sin^2 \theta}}.
\]

To abbreviate, let \( A = \sqrt{1 - e^2 \sin^2 \theta} \); \( B = \sqrt{1 - e^2} \); \( F(e, \frac{1}{2} \pi) = A \frac{1}{2} \pi \).

The variable arc \( \theta \) is named the amplitude of the function. And \( e \), being a fraction not greater than 1, is called the modulus, or eccentricity; while \( b \) is termed the complementary modulus. Also \( n \) is termed the parameter. If \( e \) is denoted by \( \sin \varphi \), and \( b \) by \( \cos \varphi \), then \( \varphi \) is sometimes called the angle of the modulus.

Examples of \( F \). From Legendre’s *Fonctions Elliptiques*, Vol. 2, Table 9:

<table>
<thead>
<tr>
<th>( e )</th>
<th>( \theta = 25^\circ )</th>
<th>( \theta = 45^\circ )</th>
<th>( \theta = 70^\circ )</th>
<th>( \theta = 90^\circ )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sin 25(^\circ)</td>
<td>0.43874 7923</td>
<td>0.79870 514</td>
<td>1.26548 460</td>
<td>1.64899 522</td>
</tr>
<tr>
<td>&quot; 45(^\circ)</td>
<td>0.44328 2328</td>
<td>0.82601 7876</td>
<td>1.36971 9477</td>
<td>1.85407 4677</td>
</tr>
<tr>
<td>&quot; 70(^\circ)</td>
<td>0.44903 50293</td>
<td>0.86652 9957</td>
<td>1.59590 6244</td>
<td>2.50455 0079</td>
</tr>
<tr>
<td>&quot; 90(^\circ)</td>
<td>0.45087 533</td>
<td>0.88137 359</td>
<td>1.73541 516</td>
<td>( \infty )</td>
</tr>
</tbody>
</table>

Values of \( E \).

| " 25\(^\circ\) | 0.44339 4028 | 0.77247 109 | 1.18039 569 | 1.49811 493 |
| " 45\(^\circ\) | 0.42957 5247 | 0.74818 6504 | 1.09900 8292 | 1.35064 3881 |
| " 70\(^\circ\) | 0.42242 64952 | 0.71714 7672 | 0.98297 5826 | 1.11837 774 |
| " 90\(^\circ\) | 0.42261 826 | 0.70710 6781 | 0.93969 2621 | 1.00000 000 |

I. Change from Algebraic to Elliptic Functions. This is generally required by the occurrence of a radical denominator of the third or fourth degree. The first step is to resolve into two quadratic factors; thus the denominator \( R = \sqrt{[A + Bx + Cx^2 + Dx^3 + Ex^4]} \)

\[
= \sqrt{[(a + bx + cz^2)(a' + b'x + c'z^2)]}.
\]

The second step is to eliminate the odd powers of the variable. For this object, one method consists in changing to \( y^2 \), after assuming

\[ a + bx + cz^2 = (a' + b'x + c'z^2)y^2. \]

Another simpler assumption is, \( x = (p + qy) + (1 + y) \). Substituting into the preceding factors, and equating each coefficient of \( y \) to zero,
These give by elimination the sum \( p+q \), and the product \( pq \); then \( p, q \) are roots of the quadratic \( u^2 - (p+q)u + pq = 0 \). Should these roots come out imaginary, Legendre proves there are two other combinations of the binomial factors \((x-r)\) etc., of \(R\), which would give real roots for \(p, q\). The two factors of \(R\) thus take the required form \(f + gy^2\).

1. When the factors of \(R\), or \((a^2 + 2ab \cos s \cdot x^2 + b^2x^4)^2\), are imaginary.

Let \(x = \sqrt{\frac{a}{b}} \tan \frac{1}{2} \theta\), \(e = \sin \frac{1}{2} \theta\); then
\[
dx = \frac{1}{R} \frac{d\theta}{2\sqrt{(ab)} \sqrt{1-e^2 \sin^2 \theta}}.
\]

2. When \(R = \sqrt{[a^2(1+b^2x^2)(1-c^2x^2)]}\). Let \(x = (1-c) \cos \theta\);
\[
\frac{b^2}{b^2 + c^2} = e^2; \quad \text{which gives} \quad \frac{dx}{R} = \frac{e}{ab} \sqrt{1-[e^2 \sin^2 \theta]}.
\]
To make the result positive, let \(\cot \theta = \sqrt{(1-e^2) \tan \theta'}\); then
\[
\frac{dx}{R} = \frac{e}{ab} \sqrt{1-[e^2 \sin^2 \theta']}.
\]

3. When \(R^2 = (1+a^2x^2)(x^2-b^2)\). Let \(x = \frac{b}{\cos \theta'}\); \(\frac{1}{1+a^2b^4} = e^2\); \(dx = \frac{ed\theta}{\sqrt{1-e^2 \sin^2 \theta'}}\).

4. When \(R^2 = (1+a^2x^2)(1+b^2x^2)\). Suppose \(a > b\); \(x = \frac{\tan \theta}{a}\); \(\frac{a^2-b^2}{a^2} = e^2\).

5. When \(R^2 = (1-a^2x^2)(1-b^2x^2)\); \(a > b\); let \(ax = \sin \theta\); \(b+a = e\); then
\[
\frac{dx}{R} = \frac{1}{a} \frac{d\theta}{\sqrt{1-e^2 \sin^2 \theta}}.
\]
The radical will be imaginary from \(x = \frac{1}{a}\) to \(\frac{1}{b}\), beyond which it becomes real to \(x = \infty\). For these real values, \(R^2 = (a^2x^4-1)(b^2x^4-1)\).

6. When \(R^2 = (a^2-x^2)(x^2-b^2)\); \(x\) lies between \(a\) and \(b\). If \(a > b\), let \(e^2 = \frac{a^2-b^2}{a^2}\), and \(x^2 = \frac{b^2}{1-e^2 \sin^2 \theta}; \frac{dx}{R} = \frac{1}{a} \frac{d\theta}{\sqrt{1-e^2 \sin^2 \theta}}\). See Legendre's treatise, Vol. I, p. 4, . . . p. 252.

7. When \(R^2 = a+bx^2+ax^4+dx^6\); let \(x^2 = z\), etc.

8. When \(R^2 = a+bx^2+ax^4\); let \(R = z\) or \(xy\), etc.

9. When \(R^2 = a+bx+ax^4+dx^6\); make \(R = x.d^4+z\), etc.

10. When \(R^2 = (a+b \sin^2 \theta)(c+d \sin^2 \theta)\); change first from \(\sin \theta\) to \(\tan \theta\).

11. When \(R^2 = A+B \cos \theta+C \sin \theta+D \cos^2 \theta+E \sin \theta \cos \theta+F \sin^2 \theta\); make \(\tan \frac{1}{2} \theta = z\), or \(\sin \theta = \frac{2z}{1+z^2}\); \(\cos \theta = \frac{1-z^2}{1+z^2}\); \(d\theta = \frac{2dz}{1+z^2}\); etc.
OTHER INTEGRAL FORMS.

12. \( R^2 = (x - a)(x - b)(x - c) \); make one factor, as \( x - a = y^2 \).

13, 14. \( \int \frac{\sin^2 \theta d\theta}{A} = \frac{1}{e^2} (F - E) \); \( \int \frac{\cos^2 \theta d\theta}{A} = \frac{1}{e^3} (E - b^2 F) \).

15, 16. \( \int \frac{\tan^2 \theta d\theta}{A} = \frac{1}{b^2} \left( A \tan \theta - E \right) \); \( \int \frac{d\theta}{A^3} = \frac{1}{b^3} E - \frac{e^2 \sin \theta \cos \theta}{b^3 A} \).

17. \( \int \frac{\sin^2 \theta d\theta}{A} = \frac{1}{b^2 e^2} \left( E - b^2 F \right) \frac{\sin \theta \cos \theta}{b^3 A} \).

18. \( \int \frac{\cos^2 \theta d\theta}{A} = \frac{1}{e^2} (F - E) + \frac{\sin \theta \cos \theta}{A} \).

19. \( \int \frac{d\theta}{A^3} = \frac{2(1+b^3)}{3 b^4} \frac{1}{3 b^2} E \frac{e^2 \sin \theta \cos \theta}{3 b^4 A} \left( 2 + 2b^2 + b^4 \right) \).

20. \( \int \frac{d\theta}{A \cos^3 \theta} = \frac{1}{b^3} \left( A \tan \theta + b^2 F - E \right) \).

21, 22. \( \int \frac{d\theta}{\cos^2 \theta} = A \tan \theta + F - E \); \( \int A \tan^2 \theta d\theta = A \tan \theta + E - 2E \).

23. \( \int A \sin^2 \theta d\theta = -\frac{1}{4} A \sin \theta \cos \theta + \frac{2e^2 - 1}{3 e^2} E + \frac{b^2}{3 e^2} F \).

24. \( \int A^2 d\theta = \frac{1}{4} e^2 A \sin \theta \cos \theta + \frac{2 + 2b^2}{3} E - \frac{b^2}{3} F \).

25. \( \int \frac{d\theta}{A \cos^3 \frac{1}{4} \theta} = 2A \tan \frac{1}{4} \theta + 2F - 2E \).

II. CONNECTION WITH THE SPHERICAL TRIANGLE. In spherical trigonometry it is proved that when one side and its opposite angle, as \( c, C \), are regarded as constant (Chauvenet's Trigonometry, p. 239), we have

\[
\frac{da}{\cos A} + \frac{db}{\cos B} = 0.
\]

The constancy of \( c, C \), makes the ratio \( e \) constant in the same triangle; thus,

\[
\sin A = e \sin a; \quad \sin B = e \sin b; \quad \sin C = e \sin c;
\]

\[
\cos A = \sqrt{1 - e^2 \sin^2 a}, \quad \cos B = \sqrt{1 - e^2 \sin^2 b}, \quad \cos C = -\sqrt{1 - e^2 \sin^2 c}.
\]

If \( a \) denote the longest side, \( C \) will denote the greatest angle, and the equation below requires that \( \cos C \) have a sign opposite to that of \( \cos A \) and \( \cos B \). Now substituting the radical values in place of \( \cos A \) and \( \cos B \) above, and integrating,

\[
\int \frac{da}{\sqrt{1 - e^2 \sin^2 a}} + \int \frac{db}{\sqrt{1 - e^2 \sin^2 b}} = \int C' \, dc.
\]

For, had the three sides been regarded as variable, the differential equation quoted above, would have a third term of the form \( C' \, dc \). As arbitrary constant here, it can contain neither \( a \) nor \( b \). When \( a = 0, b = c \); when
\[ b = 0, \ a = c; \ \text{in either case the last equation defines the form of the function} \ C'; \ \text{thus,} \]
\[ \int \frac{da}{\sqrt{1 - e^2 \sin^2 a}} + \int \frac{db}{\sqrt{1 - e^2 \sin^2 b}} = \int \frac{dc}{\sqrt{1 - e^2 \sin^2 c}}. \]

Thus the function \( F \) of the first species has the remarkable property of being expressed by two similar terms, which are not separately integrable. The relation was first discovered by the celebrated Euler in 1761.

In respect to \( E \), or the second species, we differentiate the preceding equation, clear of fractions and restore the cosines before described; giving
\[ \cos B \cos C da + \cos A \cos C db + \cos A \cos B dc = 0. \]
Substituting for each product of cosines, its value from the well known formula of spherical trigonometry,
\[ \cos A \cos B = \sin A \sin B \cos c - \cos C, \text{ or} \]
\[ = e^2 \sin a \sin b \cos c - \cos C; \]
we find
\[ \cos A da + \cos B db + \cos C dc \]
\[ = e^2 [\sin a \sin b \cos c \cos b db + \sin a \sin c \cos b db + \sin b \sin c \cos a da] \]
\[ = e^2 d(\sin a \sin b \sin c). \]
Replacing the radical values of the cosines, and integrating,
\[ \int \sqrt{1 - e^2 \sin^2 a} da + \int \sqrt{1 - e^2 \sin^2 b} db = \int \sqrt{1 - e^2 \sin^2 c} dc \]
\[ + e^2 \sin a \sin b \sin c. \]

Or
\[ E(a) + E(b) = E(c) + e^2 \sin a \sin b \sin c. \]

The equation of the Ellipse whose rectangular coordinates \( x, y \), have their origin at the center, is
\[ \frac{y^2}{B^2} + \frac{x^2}{A^2} = 1. \]

Let \( \theta \) denote an auxiliary angle; and since \( \sin^2 \theta + \cos^2 \theta = 1 \), we can make \( y = B \sin \theta, x = A \cos \theta; \ B^2 = A^2(1-e^2). \) Then the arc
\[ \sqrt{dx^2 + dy^2} = A \sqrt{1 - e^2 \sin^2 \theta} \]

The latter member, when \( A \) is unity, is identical with \( dE \); and the relation of the auxiliary arc \( \theta \) to the co-ordinates is manifest. Again if the preceding value of \( y \) is equated to that which is found from taking the focus nearest the apsis as origin, we find by comparison that \( \theta \) is the angle named in Astronomy the eccentric anomaly. And thus, if \( v \) denote the true anomaly, we have
\[ \sin \theta = \sqrt{1 - e^2} \sin v \]
\[ \frac{1 + e \cos v}{1 + e \cos v}; \]
and the preceding relation of three amplitudes or eccentric anomalies is applicable to three distances on the orbit.

The equation of the Hyperbola, whose rectangular co-ordinates have their origin at the center, is
\[ A^2 y^2 - B^2 x^2 = -A^2 B^2. \]
If we make \( A + \sqrt{A^2 + B^2} = e \); \( y = B \sqrt{1 - e^2} \tan \theta \); where \( \theta \) denotes an auxiliary angle; \( x = \sqrt{A^2 - \cos^2 \theta - A^2 \tan^2 \theta} \); then the hyperbolic arc

\[
\sqrt{dx^2 + dy^2} = B \sqrt{1 - e^2} \cdot \frac{d\theta}{\cos^2 \theta \sqrt{1 - e^2 \sin^2 \theta}} = B \sqrt{1 - e^2} \cdot dH.
\]

The Imaginary. Here let us change to \( \theta' \), and after integration, return to \( \theta \); first assuming the symmetrical relation \( \cos \theta \cos \theta' = 1 \). Then, \( \sin \theta = \sqrt{-1} \tan \theta' \); \( \sin \theta' = \sqrt{-1} \tan \theta \); \( d\theta = \sqrt{-1} (d\theta' + \cos \theta') \);

\[
dH = \frac{d\theta}{\cos^2 \theta \sqrt{1 - e^2 \sin^2 \theta}} = \frac{\sqrt{-1} \cos \theta' d\theta'}{\sqrt{1 - b^2 \sin^2 \theta'}}.
\]

Here \( b^2 = 1 - e^2 \). By division and integration this becomes,

\[
H(e, \theta) = \frac{1}{b^2} \left[ E(b, \theta') - e^2 F(b, \theta') \right].
\]

Replacing the symmetrical arcs \( \theta, \theta' \) by \( a, a' \), etc., adding the equations for \( a, b \), and subtracting for \( c \) as the greatest side of the spherical triangle, the three terms of \( F \) become zero, as before shown, while those of \( E \) with accented values change signs, and are equal to the product below;

\[
H(e, a) + H(e, b) - H(e, c) = \frac{\sqrt{-1}}{b^2} (b \sin a' \sin b' \sin c') = - \tan a \tan b \tan c.
\]

Taking \( e = \frac{1}{2}, c = 70^\circ \) and \( b = 45^\circ \), we find by the spherical formula below, \( a = 30^\circ 26' \). Then \( H(a) = 3.2782, H(b) = 1.0617, H(a) = 0.6025 \); and each member becomes \(- 1.6140 \), which verifies the preceding formula. It therefore merits our confidence. The abstruse subject of integration through imaginary limits has been systematized by Cauchy and others.

Among the earlier results, the following is readily proved by differentiation.

\[
H = \int \frac{d\theta}{\cos^2 \theta \cdot A} = \frac{A \tan \theta - E + (1 - e^2)F}{1 - e^2}.
\]

To recapitulate—when \( e \) is less than unity, let \( e \) denote the longest of the three sides \( a, b, c \) of the spherical triangle. Then will the three sides or amplitudes, and their elliptic integrals be connected by the following equations:

\[
\begin{align*}
\cos c &= \cos a \cos b + \sin a \sin b \sqrt{1 - e^2 \sin^2 c}, \\
\cos a &= \cos b \cos c + \sin b \sin c \sqrt{1 - e^2 \sin^2 a}, \\
\cos b &= \cos a \cos c + \sin a \sin c \sqrt{1 - e^2 \sin^2 b}. \\
F(a) + F(b) &= F(c), \\
E(a) + E(b) &= E(c) + e^2 \sin a \sin b \sin c, \\
H(a) + H(b) &= H(c) - \tan a \tan b \tan c, \\
II(a) + II(b) &= II(c) + \frac{n}{N} \tan^{-1} \left[ \frac{N \sin a \sin b \sin c}{1 + n \left( 1 - \cos a \cos b \cos c \right)} \right].
\end{align*}
\]
Here $N = \sqrt{n(n+1)(n+e^2)}$; and when $n$ is negative, the last term is to be changed to a logarithmic form. Similar summations might evidently be obtained for the various integrals at the end of Section I.

III. INCREASING SCALES OF MODULI. The scale discovered by Landen, and extended to the limits by Lagrange, implicitly assumes that $\sin\theta$ in one function $F$ is proportional to the derivative of $A$ in another function $F'$; that is, if $\tan \theta = \frac{\sin 2\theta'}{e + \cos 2\theta'}$, $\sin \theta = \frac{2}{1 + e} \cdot \frac{\sin \theta' \cos \theta'}{\sqrt{[1 - e^2 \sin^2 \theta']}}$;

$$e' = \frac{2\sqrt{e}}{1 + e}.$$ Or $\sin(2\theta' - \theta) = e \sin \theta; e'' = 1 - \left(\frac{1 - e}{1 + e}\right)^2$.

Or $\cos(2\theta' - \theta) = \sqrt{[1 - e^2 \sin^2 \theta]}; 2(1 + e) = e' + \sqrt{e}$.

The agreement of the fourth and sixth of these equations is proved by squaring and adding; and the agreement of the second equation with the first can be shown by squaring the first, then changing from $\sin \theta$ to $\cot \theta$, and eliminating $\cot \theta$, which leaves an identical equation.

Let us here introduce the auxiliary angle such that $\sin \varphi = e$, $\sin \varphi' = e'$, ... For computing the ascending scale;

$$e' = \frac{2\sqrt{e}}{1 + e}; e'' = \frac{2\sqrt{e'}}{1 + e'}; e''' = \frac{2\sqrt{e'}}{1 + e'}; \ldots$$

or $\tan \frac{\varphi}{e} = \sqrt{\sin \varphi}$; $\tan \frac{\varphi'}{e} = \sqrt{\sin \varphi'}$; $\tan \frac{\varphi''}{e} = \sqrt{\sin \varphi''}$; to be continued till the value of $e$ is sufficiently near to the limit 1, which renders $F$ an exact integral, as far as required.

For computing the series of amplitudes, we have

$$\sin(2\theta' - \theta) = e \sin \theta = \sin \varphi \sin \theta;$$
$$\sin(2\theta'' - \theta) = e' \sin \theta' = \sin \varphi' \sin \theta';$$
$$\sin(2\theta''' - \theta) = e'' \sin \theta'' = \sin \varphi'' \sin \theta''.$$ Expanding $\sin(2\theta' - \theta)$, and dividing the equation by $\sin \theta$, we find

$$\cot \theta = \frac{e + \cos 2\theta'}{\sin 2\theta'}; \quad d \cot \theta = \frac{-(1 + e \cos 2\theta') d \theta'}{\sin^2 2\theta'}.$$

Now

$$dF = \frac{d\theta}{A} = \frac{-\sin^2 \theta d \cot \theta}{A} = \frac{-\sin \theta d \cot \theta}{\sqrt{[1 + \cot^2 \theta - e^2]}}.$$ Substituting for $\sin \theta$ its value from the assumed equation above, and for $\cot \theta$ and $d \cot \theta$, the values just found, we have

$$dF = \frac{d\theta}{\sqrt{[1 - e^2 \sin^2 \theta]}} = \frac{2}{1 + e} \cdot \frac{d\theta'}{\sqrt{[1 - e^2 \sin^2 \theta']}}.$$ When the scale $e$, $e'$, $e''$, ... has been extended so near to the limit 1 that the last modulus may be taken as 1, let $\theta$ denote the corresponding amplitude, which is also a limit; then

$$F(1, \theta) = \int \frac{d\theta}{\sqrt{[1 - \cos^2 \theta]}} = \int \frac{d\theta}{\cos \theta} = \log \tan(45^\circ + \frac{1}{2} \theta).$$
As the moduli increase, the amplitudes diminish to this limit; thus,

\[ F(e, \theta) = \frac{2}{1+e^2} F(e', \theta') = \frac{2}{1+e^2} F(e'', \theta''), \ldots \]

\[ F(e, \theta) = \log \tan (45^\circ + \frac{1}{2} \theta), \frac{2}{1+e^2} \frac{2}{1+e^2} \frac{2}{1+e^2} \ldots 1. \]

When \( \theta = 90^\circ \), the first equation of amplitude reduces to \( \sin (2\theta' - \theta) = e \sin \phi \), or \( \theta' = 45^\circ + \frac{1}{2} \phi \).

Let \( M = \frac{2}{1+e^2} \frac{2}{1+e^2} \frac{2}{1+e^2} \ldots 1 = \sqrt{\frac{e^2 e'' e''' \ldots 1}{e}} = \frac{\tan \frac{1}{2} \phi \cdot \tan \frac{1}{2} \phi \ldots 1}{\tan \frac{1}{2} \phi}. \]

\[ F(e, \theta) = M \log \tan (45^\circ + \frac{1}{2} \theta). \]

The Napierian logarithm here indicated, is found by multiplying the common logarithm by 2.302585093.

Preparatory to the next section, the difference of the two equations, \( \sin \theta = \sin \theta_1 \) and \( \sin (2\theta' - \theta) = e \sin \phi \), divided by their sum, gives \( \tan (\theta - \theta') = \tan \theta' = \sqrt{[1-e^2 \sin^2 \theta]} \).

Note. The relation discovered by Gauss in 1818 gives the same moduli as before, but a different series of amplitudes; thus, \( e' = 2\sqrt{e^2 + (1+e)} \),

\[ \sin \theta' = \frac{(1+e) \sin \theta}{1+e \sin^2 \theta}; \frac{d\theta}{\sqrt{[1-e^2 \sin^2 \theta]} = \frac{1}{1+e} \frac{d\theta'}{\sqrt{1-e^2 \sin^2 \theta}}}. \]

This scale of Gauss can be derived from that of Lagrange by changing from \( \theta \) and \( \theta' \) to \( \mathcal{F} \) and \( \mathcal{F}' \) after assuming \( \cos \theta \cos \mathcal{F} = 1, \cos \theta' \cos \mathcal{F'} = 1 \), as in Section II.

IV. DECREASING SCALES OF MODULI. The preceding scale of moduli can be prolonged in the opposite direction, tending to the limit 0; so that the entire scale will be 0, \( \ldots e^{\infty}, e^{\infty}, e, e', e'', \ldots \), 1. Therefore reversing the accents in the above equation, we have for computing the decreasing series of moduli, as well as the amplitudes:

\[ \frac{1-e}{1+e} = \sqrt{[1-e^2]} = b; \frac{e}{1+e^2} = \frac{1-b}{1+b}; \frac{e^{\infty}}{1+e^{\infty}} = \frac{1-b^{\infty}}{1+b^{\infty}}; \frac{e^{\infty}}{1+e^{\infty}} = \frac{1-b^{\infty}}{1+b^{\infty}}; \ldots \]

\[ e = \sin \phi, e' = \sin \phi^0, \ldots; e'' = \tan^2 \frac{1}{2} \phi; e^{\infty} = \tan^2 \frac{1}{2} \phi^0; \ldots \]

\[ e = \sqrt{[1-b^2]} = \frac{1-b}{1+b}; \frac{b}{1+b} = \frac{2 \sqrt{b}}{1+b} = \frac{b^0}{\sqrt{b}}. \]

\[ \tan (\theta' - \theta) = b \tan \theta = \cos \phi \cdot \tan \theta; \]

\[ \tan (\theta^{\infty} - \theta^0) = b^0 \tan \theta^0 = \cos \phi^0 \cdot \tan \theta^0, \text{ etc.} \]

\[ F(e, \theta) = \frac{1+e}{2}. F(e', \theta') = \frac{1+e}{2}. \frac{1+e^{\infty}}{2}. F(e^{\infty}, \theta^{\infty}), \ldots \]

(To be continued.)
SUPPOSE $\varphi x$ and $\varphi' x$ are finite and continuous functions for all values of $x$ from $x = x'$ to $x = x' + h$.

Then

$$\frac{\varphi(x' + h) - \varphi x'}{h} = \varphi'(x' + \psi h),$$

where $\psi$ is a positive proper fraction.

Let $A$ and $B$ be the algebraically greatest and least values of $\varphi' x$ between the values $x'$ and $x' + h$ of $x$. Then $A, B, \varphi' x$ and $\frac{\varphi(x' + h) - \varphi x'}{h}$ are all positive, or all negative accordingly as $\varphi x$ is an increasing or a decreasing function.

Put

$$y = Ax - \varphi x \quad (1), \quad z = \varphi x - Bx. \quad (2)$$

Then

$$\frac{dy}{dx} = A - \varphi' x, \quad \text{and} \quad \frac{dz}{dx} = \varphi' x - B, \text{both of which are always positive, and consequently } y \text{ and } z \text{ are always increasing functions.}$$

Let $x$ take each of the two values $x'$ and $x' + h$ in (1) and (2). Then,

$$y' = A(x' + h) - \varphi(x' + h), \quad z' = \varphi(x' + h) - B(x' + h),$$

$$y = Ax' - \varphi x', \quad z = \varphi x' - Bx'. \text{ Therefore}$$

$$\frac{y' - y}{h} = A - \frac{\varphi(x' + h) - \varphi x'}{h}, \quad \frac{z' - z}{h} = \frac{\varphi(x' + h) - \varphi x'}{h} - B.$$

Because $y$ and $z$ are increasing functions, $y' - y$ and $z' - z$ are of the same sign as $h$, and therefore the members of the last two equations are always positive.

Consequently $[\varphi(x' + h) - \varphi x'] + h$ is always less than $A$ and greater than $B$. But $\varphi' x$ in passing from $A$ to $B$, passes through all intermediate values, and therefore through $[\varphi(x' + h) - \varphi x'] + h$; and in this value of $\varphi' x$, $x$ has some value between $x'$ and $x' + h$. Let $x' + \psi h$ represent this value of $x$, $-\psi$ being a positive proper fraction. Then, $[\varphi(x' + h) - \varphi x'] + h = \varphi'(x' + \psi h)$; or we can write $[\varphi(x + h) - \varphi x] + h = \varphi'(x + \psi h)$, which will be true for all values of $x$ and $h$ within the limits of finite and continuous values of $\varphi x$ and $\varphi' x$.

Suppose now that $\varphi x, \varphi' x, \varphi'' x \ldots \varphi^n x$ are finite and continuous functions, for all values of the variable from $x$ to $x + h$; then by the foregoing principle we can have the following equations:

$$[\varphi (x + h) - \varphi x] + h = \varphi'(x + \psi_1 h), \text{ where } \psi_1 \text{ is between } 0 \text{ and } 1. \quad (a)$$

$$[\varphi'(x + \psi_1 h) - \varphi x'] + \psi_1 h = \varphi''(x + \psi_2 h), \quad \psi_2 \text{ " } 0 \text{ " } \psi_1. \quad (b)$$

$$[\varphi''(x + \psi_2 h) - \varphi' x] + \psi_2 h = \varphi'''(x + \psi_3 h), \quad \psi_3 \text{ " } 0 \text{ " } \psi_2. \quad (c)$$
\[ \varphi''(x + \phi_3 h) - \varphi''(x) = \varphi''(x + \phi_4 h), \text{ where } \phi_4 \text{ is } 0 \text{ and } \phi_3. \]  
\[ \varphi^{n-1}(x + \phi_{n-1} h) - \varphi^{n-1}(x) = \varphi^n(x + \phi_n h), \quad \phi_n = \phi_1 = 0 \text{ and } \phi_{n-1}. \]
\[ \varphi^{2n-1}(x + \phi_{2n-1} h) - \varphi^{2n-1}(x) = \varphi^{2n}(x + \phi_{2n} h), \quad \phi_{2n} = \phi_2 = 0 \text{ and } \phi_{2n-1}. \]

In general, \( \phi_1, \phi_2, \ldots \) are functions of both \( x \) and \( h \), and do not reduce to zero when \( h = 0 \). For each being always positive, and never exceeding unity, neither can contain \( h \) as a factor in every term and consequently cannot become zero when \( h = 0 \). Let \( \phi_1', \phi_2', \phi_3', \ldots \) be the values to which \( \phi_1, \phi_2, \phi_3, \ldots \) respectively reduce, when \( h = 0 \).

In (b), replace \( \phi_1 \) in the denominator of the left member by \( \phi_1' \), and denote what the right member becomes by \( \varphi'(x + \phi_1' h) \). Then \[ \varphi'(x + \phi_1 h) = \varphi''(x + \phi_1' h). \]

Let \( \phi_2' \) be substituted for \( \phi_2 \) in the left member of (c), and denote what the right member becomes by \( \varphi'''(x + \phi_2' h) \).

Then \[ \varphi''(x + \phi_2' h) - \varphi''(x) = \varphi'''(x + \phi_2' h). \]

In this, let \( \phi_3' \) be substituted for \( \phi_2' \) in the denominator of the left member and denote what the right member becomes by \( \varphi''(x + \phi_3' h) \). Then \[ \varphi'(x + \phi_2' h) - \varphi'(x + \phi_2' h) = \varphi''(x + \phi_3' h). \]

Continuing this process, we shall finally get \[ \varphi^{n-1}(x + \phi_{n-1} h) - \varphi^{n-1}(x) = \varphi^n(x + \phi_n h). \]

Collecting the several results, we have with (a), \[ \varphi(x + h) = \varphi(x + \phi_1 h), \quad \varphi'(x + \phi_1 h), \quad \varphi''(x + \phi_2 h), \quad \ldots \quad \varphi^{n-1}(x + \phi_{n-1} h) = \varphi^n(x + \phi_n h). \]

Eliminating \( \varphi'(x + \phi_1 h), \varphi'(x + \phi_2 h), \ldots \varphi^{n-1}(x + \phi_{n-1} h) \), we get \[ \varphi(x + h) = \varphi(x + h) \times \varphi''(x + \phi_2 h) \times \varphi''(x + \phi_3 h) \times \varphi''(x + \phi_4 h) \times \ldots \times \varphi''(x + \phi_n h) \times \varphi^n(x + \phi_n h). \]

In (3), \( \phi_1', \phi_2', \ldots \phi_{n-1}' \) are independent of \( h \), and \( \phi_n' \) is some quantity which is in general a function of both \( x \) and \( h \). By differentiating (3) \( n \) times with respect to \( h \), we obtain the following equations:

\[ \varphi'(x + h) = \varphi'(x + h) + \varphi''(x + h) + \varphi''(x + h) \times \varphi''(x + h) \times \varphi''(x + h) \times \ldots \times \varphi''(x + h) \times \varphi''(x + h), \]

\[ \varphi''(x + h) = 2\varphi_1' \varphi' + 4\varphi_2' \varphi' + 4\varphi_3' \varphi_2' \varphi' + \ldots \times \varphi''(x + h). \]

\[ \varphi'''(x + h) = 3\varphi_1' \varphi_2' \varphi' + 3\varphi_2' \varphi_3' \varphi_2' \varphi' + \ldots \times \varphi'''(x + h). \]

\[ \varphi^{n-1}(x + h) = (n-1)! \varphi^{n-1}(x + h) \times \varphi''(x + h) \times \varphi''(x + h) \times \ldots \times \varphi''(x + h), \]

\[ \varphi^n(x + h) = 1.2.3 \times \varphi^{n-1}(x + h) + 2.3 \times \varphi^{n-1}(x + h) + \ldots \times \varphi^n(x + h). \]

In these equations, let \( h = 0 \). Then we obtain \( \varphi x = \varphi x; \varphi'' x = 2\varphi_1' \varphi x. \)

\[ \varphi_1' = \frac{1}{2}. \varphi x = 2.3 \times \varphi_2' x, \quad \varphi_2' = \frac{1}{3}. \varphi x = 2.3.4 \varphi_3' x. \]
\[ \phi_3 = \frac{1}{2}; \text{ and in general, } \phi_1 = 1 + n. \text{ These values of } \phi_1 \ldots \phi_{n-1} \text{ in (3) and (4) give respectively:} \]

\[
\varphi(x+h) = \varphi x + h\varphi' x + \frac{h^2}{2}\varphi'' x + \ldots + \frac{h^n}{n!}\varphi^n(x + \phi'_n h). \tag{5}
\]

\[
\varphi^n(x+h) = \varphi^n(x + \phi'_n h) + nh \frac{d\varphi^n(x + \phi'_n h)}{dh} + \ldots \tag{6}
\]

From (6) we find that \( \varphi^n(x + \phi'_n h) < \varphi^n(x+h). \) \( \ldots \phi''_n < 1. \) If \( m \) be any integer between \( n \) and \( 2n, \) then (5) can become

\[
\varphi(x+h) = \varphi x + h\varphi' x + \frac{h^2}{2}\varphi'' x + \ldots + \frac{h^n}{n!}\varphi^n x + \ldots + \frac{h^m}{m!}\varphi^m(x + \phi'_m h). \tag{7}
\]

\[
\ldots \phi^n(x + \phi'_m h) = \varphi^n x + \frac{h}{n+1}\varphi^{n+1} x + \ldots + \frac{h^m}{(n+1)(n+2)}\varphi^{m+2} x + \ldots + \frac{h^n}{(n+1)(n+2)\ldots m} \varphi^n(x + \phi'_m h). \tag{7}
\]

\[
\ldots \varphi^n(x + \phi'_m h) > \varphi^n x, \text{ and } \phi''_m \text{ is positive; but } \phi''_m < 1, \text{ therefore it is a positive proper fraction. From (5) and (7) we also find that } \phi''_m > \phi'_m. \text{ Hence we have } 1 > \phi'_1 > \phi'_2 > \phi'_3 > \phi'_4 \ldots \phi''_m > \phi'_m. \text{ Hence also, we can make } \phi''_m \text{ as small as we please, } provided \text{ we can make } n \text{ large enough and still have } \varphi^n x \text{ finite and continuous.}
\]

With this condition, we can make \( \varphi^n(x + \phi'_m h) \) differ from \( \varphi^n x \) by as small an amount as may be desired, and we can write:

\[
\varphi(x+h) = \varphi x + h\varphi' x + \frac{h^2}{2}\varphi'' x + \ldots + \frac{h^n}{n!}\varphi^n x, \tag{8}
\]

which gives the value of \( \varphi(x+h) \) to any desired degree of accuracy when \( n \) is large enough.

Taylor's formula can be derived directly from equations (a), (b), (c), \ldots as follows:—Because \( 1 > \phi'_1 > \phi'_2 > \ldots > \phi'_m > \phi'_n, \) we can make \( \varphi^n(x + \phi'_m h) \) differ from \( \varphi^n x \) by as small an amount as we please, \( provided \) we can make \( n \) large enough and still have \( \varphi^n x \) finite and continuous. Hence eliminating \( \varphi'(x+\phi'_1 h), \varphi''(x+\phi'_2 h), \ldots \) we get

\[
\varphi(x+h) = \varphi x + h\varphi' x + \frac{h^2}{2}\varphi'' x + \varphi_2 h^3\varphi''' x + \ldots + \varphi_{n-1} h^n\varphi^n x + \varphi_1 \varphi_2 \ldots \varphi_{n-1} h^n\varphi^n x + \varphi_2 h^2\varphi'' x + \varphi_1 \varphi_2 h^2\varphi'' x + \ldots + \varphi_1 \varphi_3 h^2\varphi'' x + \ldots \varphi_2 h^2\varphi'' x, \tag{9}
\]

where the values of \( \varphi(x+h), \) will be expressed to any desired degree of accuracy, when \( n \) is large enough.

Forming the differential coefficients of (9), regarding \( h \) as constant; and again regarding \( x \) as constant, and subtracting the second result from the first, we obtain,

\[
0 = (1-2\phi_1)h\varphi'' x + \left( \frac{d\phi_1}{dx} - \frac{d\phi_1}{dh} \right) h^2\varphi''' x + \left( \frac{d\phi_1}{dx} - \frac{d\phi_1}{dh} \right) \varphi_2 h^3\varphi'' x + \left( \frac{d\phi_1}{dx} - \frac{d\phi_1}{dh} \right) \varphi_1 \varphi_2 h^3\varphi''' x + \ldots \tag{10}
\]
If in (10) we put \( 1 - 2\phi_1 = 0, \phi_1 - 3\phi_1\phi_2 = 0, \phi_1\phi_2 - 4\phi_1\phi_2\phi_3 = 0, \ldots \) then, \( \phi_1 = \frac{1}{2}, \phi_2 = \frac{1}{3}, \phi_3 = \frac{1}{4}, \ldots \phi_{2n-1} = 1 + 2n; \) and these values being constants,

\[
\frac{d\phi_1}{dx}, \frac{d\phi_1}{dh}, \frac{d\phi_2}{dx}, \frac{d\phi_2}{dh}, \ldots
\]

each becomes zero, and equation (10) is completely satisfied. Hence, \( \phi_1 = \frac{1}{2}, \phi_2 = \frac{1}{3}, \phi_3 = \frac{1}{4}, \ldots \phi_{2n-1} = 1 + 2n, \) are always included among the possible values of these fractions in (9), and give

\[
\varphi(x+h) = \varphi x + h \varphi' x + \frac{h^2}{2} \varphi'' x + \ldots + \frac{h^n}{n!} \varphi^n x + \frac{h^{n+1}}{(n+1)!} \varphi^{n+1} x + \ldots + \frac{h^{2n}}{2^n} \varphi^{2n} x + \ldots \tag{11}
\]

We can also have from (a), (b), (c), 

\[
\varphi(x+h) = \varphi x + h \varphi' x + \varphi_1 h^2 \varphi'' x + \varphi_1 \varphi_2 h^3 \varphi''' x + \varphi_1 \varphi_2 \ldots \varphi_{2n-1} h^n 
\]

\[
\times \varphi^n(x+\varphi h)
\]
in which, giving \( \varphi_1, \varphi_2 \ldots \) their foregoing values and denoting by \( \varphi'' \) what \( \varphi \) may in consequence become, we get equation (5) in which \( \varphi'' \) has been shown to be a positive proper fraction.

NOTE ON THE HISTORY OF THE METHOD OF LEAST SQUARES.

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I.

In the March number of this Journal I stated the first proof of the Method of Least Squares to have been given by Dr. Robert Adrain in 1808 and the second by Gauss in 1809. At that time I had not seen Adrain’s original paper and my information concerning it had been entirely derived from an account and partial reprint given by Prof. Cleveland Abbe in the Amer. Jour. Sci., 1871, Vol. I, pp. 411–415. Having lately been able to consult a copy of The Analyst or Mathematical Magazine in which Adrain’s paper was published I found that on pages 96 and 97, to which Prof. Abbe makes no reference, there is given a second deduction of the law of facility of error of an entirely different nature from that presented on pages 93–95. As this is a matter of considerable historical interest and as The Analyst for 1808 is quite rare I give the proof in Adrain’s own words,
“Suppose that the length and bearing of $AB$ are to be measured; and that the little equal straight lines $Bb$, $BC$ are the equal probable errors, the one $Bb = Bb'$ of the length of $AB$, and the other $Be = Be'$ (perpendicular to the former) of the angle at $A$, when measured on a circular arc to the radius $AB$: and let the question be to find such a curve passing through the four points $b$, $c$, $b'$, $c'$, which are equally distant from $B$, that, supposing the measurement to commence at $A$, the probability of terminating on any point of the curve may be the same as the probability of terminating on any one of the four points $b$, $c$, $b'$, $c'$.”

The reasoning following this is long and trivial and ends by concluding that “the curve must be the simplest possible” and that it “must consequently be the circumference of a circle having its centre in $B$.” This established, the exponential law of facility is deduced as follows:

“Now let us investigate the probability of the error $Bm = x$, and of $mn = y$. Let $X$ and $Y$ be two similar functions of $x$ and $y$ denoting these probabilities, $X'$, $Y'$ their logarithms, then $XY = \text{constant}$, or $X' + Y' = \text{constant}$, and therefore $X' + Y' = 0$, or $X'' - x + Y'' - y = 0$, whence $X'' = -Y''$. But $x^2 + y^2 = r^2 = BB^2$ therefore $xx = -yy$, by which dividing $X' = -Y'$, we have $X'' = x = Y'' = y$; and therefore, by a fundamental principle of similar functions, the similar functions $X'' = x$ and $Y'' = y$ must be each a constant quantity: put then $X'' = x = n$, and we have $X'' = nx$, that is $X'' = nx^2$, and the fluent is $X'' = c + \frac{1}{2}nx^2$; in like manner we find $Y'' = c + \frac{1}{2}ny^2$, and therefore the probabilities themselves are $e^{c + \frac{1}{2}nx^2}$ and $e^{c + \frac{1}{2}ny^2}$, in which $n$ ought to be negative, for the probability of $x$ grows less as $x$ grows greater.”

This proof is the same as that given in 1850 by Herschel (J. F. W.) and usually called Herschel’s proof. It is defective in taking $XY = \text{constant}$, that is, in regarding the rectangular deflections $x$ and $y$ as independent.

It appears then that to Dr. Adrain the second as well as the first proof of the Method of Least Squares must be credited. Why Prof. Abbe, in his otherwise exhaustive account of Adrain’s paper, should have overlooked this demonstration is hard to understand, unless we suppose that he accidentally turned two leaves of *The Analyst* at once and thus jumped from the bottom of page 95 to the top of page 98.
II.

I have lately spent (and perhaps wasted) a good deal of labor in making a catalogue or list of all writings relating to the Method of Least Squares and the Theory of accidental Errors of Observation. Books which devote only a page or two to the subject and practical papers in which the Method is used briefly and incidentally I have not cared to include. My aim has been to record every book or paper which can be regarded as a contribution to the science of the Adjustment of Observations, whether written from a theoretical or practical standpoint. A few statistics from this catalogue may perhaps be properly stated here.

The whole number of writings recorded is 404; of these 94 may be classed as books and 310 as memoirs or papers. The earliest bears the date 1722 and the latest 1876. Previous to 1805, the year of Legendre’s announcement of the principle of Least Squares there are 22 titles; since 1805 the numbers published in the several decades are

from 1805 to 1814, 18 titles,
1815  1824, 29
1825  1834, 33
1835  1844, 45
1845  1854, 63
1855  1864, 71
1865  1874, 94

showing a steady increase in the interest and importance of the subject.

These books and memoirs are in eight languages, and classified according to the place of publication they fall under twelve countries. The following table shows this classification:

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<thead>
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<th>Languages</th>
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<td>Great Britian</td>
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Total . . . . . . 404 . . . . . . . . . 404
These works were written by 192 different authors, 125 of whom produced only one book or paper each. The largest number written by one author is 14; this author was Laplace. Out of the 404 works I have seen and actually inspected 309.

Like all bibliographical researches of this kind my list cannot be regarded as complete. With better library facilities the number of titles in the Italian, Dutch and Scandinavian languages would be much increased; and one who can consult the Russian and Hungarian literature might undoubtedly find a few titles to add although the articles themselves would probably not be of great value. I hope at some future time to be able to publish the list accompanied with historical and critical notes giving brief accounts of the contents of each book or memoir with perhaps an estimate of its value.

PEDAL CURVES.

BY PROF. W. W. JOHNSON, ANNAPOlis, MARYLAND.

When the rolling curve is equal to the fixed curve, the curves touching at corresponding points, as in the Figure on page 71, No. 3, Analyst, the locus of any point connected with the rolling curve is the same as the Pedal of the given curve, doubled in linear dimensions. For consider the point in the plane of the fixed curve corresponding to the generating point, as $D$ in the figure; it is plain that $D$ and $D''$ are symmetrical points with reference to the common tangent at $P$, so that $DD''$ is perpendicular to the tangent and bisected by it, and the locus of $D''$ is similar to that of the foot of the perpendicular upon the tangent, 2 being the ratio, and $D$ the centre of similitude.

It may be remarked that since the tangent to the roulette, or locus of $D''$, and to the pedal are parallel, the well known and obvious property of roulettes that $PD''$ is a normal affords a simple proof of the property of the pedal that the inclination of its tangent and radius vector is the same as the corresponding angle at the point of contact $P$ in the original curve.

When the equation of the tangent to the given curve, in terms of its direction ratio $m$, is known, the equation of the pedal is very easily found; for, $(h, k)$ being the pedal origin, the equation of the perpendicular upon the tangent is

$$y - k = - \frac{1}{m} (x - h),$$

hence if we eliminate $m$ between this equation and that of the tangent, by substituting in the latter $m = (h - x)/(y - k)$, we shall at once have the equation of the pedal.
Thus for the parabola \( y^2 = 4ax \), the tangent being \( y = mx + (a + w) \) the pedal is

\[
y + x \frac{x}{y} = h + a \frac{y}{x} = k = 0.
\]

The pedal origin satisfies the equation of the pedal even when it is not a point of the geometrical locus. The following explanation of this circumstance is taken from Briot and Bouquet’s Ge’ome’trie Analytique. When a locus is determined by the intersection of two variable lines, \( A \) and \( B \), whose equations are

\[
f_1(x, y, a) = 0 \quad \text{and} \quad f_2(x, y, a) = 0,
\]

containing one arbitrary parameter \( a \) (or \( n \) parameters connected by \( n - 1 \) relations) the locus

\[
f(x, y) = 0
\]

is found by eliminating \( a \). Now if the line \( A \) always passes through the fixed point \( (x_1, y_1) \) this point will satisfy \( f(x, y) = 0 \); for substitute \( x_1 \) and \( y_1 \) for \( x \) and \( y \) in the equation of \( B \), we may determine from the result certain real or imaginary values of \( a \); let \( a_1 \) be one of these values, then since, by hypothesis \( (x_1, y_1) \) satisfies the equation of \( A \), no matter what the value of \( a \), the values \( x_1, y_1 \) and \( a_1 \) satisfy the equations \( f_1 = 0 \) and \( f_2 = 0 \) and hence also \( f = 0 \) which is derived from them.

In the case of the pedal the fixed point is the pedal origin and since at least two tangents real or imaginary can be drawn through it, it is always a node. When the pedal origin is a focus however the equation breaks up into two factors, one of which denotes an infinitesimal circle at the pedal origin, hence this factor may be removed and the other represents only the geometrical locus which is a circle.

The following important proposition connects pedals with inverse curves:- If we take the polar reciprocal of a given curve with respect to a circle, the pedal to the given curve, with respect to the centre of the auxiliary circle, is the inverse of the reciprocal curve with respect to the same point. This is evident geometrically as follows; let \( O \) denote the pedal origin and \( OP' \) the perpendicular upon the tangent to the given curve, then the pole of the tangent with respect to a circle whose radius is \( k \) is a point \( P_1 \) so taken on the perpendicular \( OP' \) that \( OP_1 \cdot OP' = k^2 \). The polar reciprocal is the locus of \( P_1 \) and the pedal is the locus of \( P' \), but by the definition of inversion, these loci are mutually inverse curves.

The same thing may be proved analytically as follows, taking the origin of coordinates at the pedal origin, the polar of \( (x_1, y_1) \) with respect to the circle \( x^2 + y^2 = k^2 \) is

\[
x x_1 + y y_1 = k^2,
\]
hence if the tangent to the given curve be put in this form, we can express $x_1$ and $y_1$ in terms of an arbitrary parameter and, by eliminating this parameter, find the equation of the locus of $(x_1, y_1)$, which is the reciprocal curve. Now if $(x', y')$ describes the inverse of this locus we have (as at p. 43, Analyst, Vol. IV.)

$$x_1 = \frac{k^2 x'}{x'^2 + y'^2} \quad \text{and} \quad y_1 = \frac{k^2 y'}{x'^2 + y'^2},$$

whence the tangent line may be written in the form

$$xx' + yy' = x'^2 + y'^2,$$

but this is the equation of a line passing through $(x', y')$ and perpendicular to the line joining this point to the origin; that is, $(x', y')$ is the foot of the perpendicular upon the tangent and describes the pedal to the given curve.

The reciprocal of a conic being a conic, it follows that the pedals to a conic are the inverse curves of another conic, that is they are either circles or (see p. 47, Analyst, Vol. IV.) the nodal forms of the bicuspid quartic and circular cubic. In particular, the central pedals and central inverse curves of the conic consist of the same class of curves, namely the symmetrical bicuspid quartics with node at the centre of symmetry.

In the case of the rectangular hyperbola $x' - y' = a^2$, the reciprocal with respect to the circle $x^2 + y^2 = a^2$ is identical with the given curve; for putting the equation of the tangent in the form

$$xx_2 - yy_2 = a^2, \quad (1)$$

where

$$x_2^2 - y_2^2 = a^2, \quad (2)$$

and comparing with the equation of the polar of $(x_1, y_1)$

$$xx_1 + yy_1 = a^2, \quad (3)$$

we have

$$x_2 = x_1 \quad \text{and} \quad y_2 = -y_1,$$

and, substituting in (2), we have

$$x_1^2 - y_1^2 = a^2$$

for the equation of the reciprocal. Hence the central pedal of the rectangular hyperbola is the same as its central inverse, namely the Lemniscate. The locus of the centre of a rectangular hyperbola rolling upon an equal hyperbola is therefore a lemniscate with axis double that of the hyperbola.

The theorem that the reciprocal of a conic with respect to a focus is a circle may be derived from the familiar property that the pedal of a conic with respect to a focus is a circle (or straight line in case of the parabola); for the inverse of the pedal is the reciprocal, and the inverse of a circle is known to be a circle.

The pedal to the circle is the limacon; for, taking the pedal origin as pole, and the centre of the circle (radius $a$) at the distance $b$ on the initial line,
the equation of the pedal is at once seen to be
\[ r = a + b \cos \theta, \]
which is the polar equation of the limacon. Accordingly, the limacon is
the inverse of the reciprocal of the circle, that is, it is the inverse of a conic
with respect to a focus.

The reciprocal of a conic with respect to a point on the curve is a para-
bola; with respect to a point outside the curve (that is one from which real
tangents can be drawn) the reciprocal is an hyperbola; and for a point
within the curve, it is an ellipse.

Accordingly, the pedal with respect to a point on the curve, like the in-
verse of the parabola, is cuspidal; for a point without, it is crunodal, like
the inverse of an hyperbola; and for a point within, it is acnodal, like the
inverse of an ellipse. On the other hand, the pedal to the parabola, like
the inverse with respect to a point on the curve, is a cubic, while the pedals
of the other conics are quartics.

THE DIFFERENTIAL EQUATIONS OF PROBLEM 165.

BY PROF. A. HALL, NAVAL OBS., WASHINGTON, D. C.

Since the differential equations of this problem are those which furnish
the theory of the pure elliptic motion of the planets they have been very
completely discussed, and it seems worth while to notice a few of the prop-
eries that have been deduced. First, we may show without any integra-
tion that the planet moves in a plane. If we consider three successive po-
sitions of the planet whose coordinates are \( x, x + dx, \) and \( x + 2dx + d^2x, \)
with similar values for \( y \) and \( z; \) and form the determinant of the third order
whose elements are these coordinates, by means of the given differential equa-
tions this determinant reduces to zero, which is the condition that the three
points are in a plane.

One of the most elegant discussions of the equations is that given by
Laplace, Mec. Cel., Book II. arts. 17 and 18. Laplace first notices that
each of the equations may be written in the form
\[ d. \left( r^3 \frac{d^2 x}{dt^2} \right) + m. \ dx = 0, \]
and then shows that we have also the similar form
\[ d. \left( r^3 \frac{d^2 r}{dt^2} \right) + m. \ dr = 0. \] (1)
From this last form he deduces the linear equation for \( r \),

\[
r = \frac{h^2}{m} + \lambda x + \gamma y,
\]

\( h, \lambda \) and \( \gamma \) being constants. In his Theoria Motus, art. 3, Gauss uses this equation in the form

\[
r + ax + by = c,
\]

and has the remark "et quidem \( \gamma \) quantitatem natura sua semper positivam;" which from Laplace's form is evidently correct.

If we integrate equation (1) we have

\[
\frac{d^2 r}{dt^2} + \frac{m}{r^2} + \frac{k}{r^3} = 0,
\]

the constant \( k \) having a value different from zero. Hesse gives the following interpretation to this equation: If one lived on the radius vector of a planet he would not be able to explain the motion of the planet by means of the law of gravitation, but must regard the inverse third power of the distance.

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**TWELVE ORIGINAL PROBLEMS.**

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**BY PLINY EARLE CHASE, LL. D., PROF. OF PHILOS. IN HAVERFORD COL.**

1. Let \( f \) represent any central force whatever, varying inversely as the square of the distance, \( r \) being the radius of a perpetual circular oscillation produced by the force. Required the mean velocity of a synchronous radial oscillation.

2. If material particles under the influence of \( f \) were constrained to move in various orbits, (circular, elliptic, parabolic, and rectilinear,) collisions near the centre would produce a nucleus, which would rotate, on account of the resultant of such portions of the orbital forces as were not otherwise represented. Required the law of varying velocity, in terms of \( r \), under nuclear contraction or expansion.

3. If a viscous nucleus were accompanied by a non-viscous, elastic, atmosphere, rotating with the nucleus on account of central pressure, what formula would express the limit of possible atmosphere, in terms of \( f, r \), and \( v \), \( r \) being the nuclear radius and \( v \) the velocity of equatorial rotation?

4. If such a nucleus and atmosphere are simultaneously expanding or contracting, express the varying ratio of the nuclear radius \( (r) \) in terms of the atmospheric radius \( (r') \).

5. Give an expression for the common tangential velocity towards or from which the circular-orbital velocity \( (v') \) and the equatorial-rotation velocity \( (v) \) both tend.
6. Give an expression for the common tangential velocity, towards which the parabolic-perifocal, or dissociating velocity \( (v' \sqrt{2}) \) and the equatorial-rotation velocity both tend.

7. Give an expression for the common velocity towards which the dissociating velocity \( (v' \sqrt{2}) \) and the mean radial velocity (Prob. 1) both tend.

8. Required the values of the common velocities in the Solar System, (probs. 5, 6, 7,) estimating the time of solar rotation at 25.4 days.

9. Required the value of the ultimate velocity (Prob. 7) for the largest planets in the intra-asteroidal and the extra-asteroidal belt (the Earth and Jupiter), estimating Jupiter's day at 9.6 h.

10. Granting the postulates of Problems 8 and 9, what is the sun's mean distance from the earth?

11. What would be the ratio of elasticity to density, in any medium which would admit of the least velocity assigned to gravity by Laplace, (100,000,000 times the velocity of light), the ratio in air being assumed as unity?

12. What must be the nature of a medium which would admit of an instantaneous velocity, such as Laplace supposed the velocity of gravity to be?—[To the foregoing probs. Prof. Chase appends the following answers.]

1. \( t = 2\pi v/(f+r) \); \( 4\pi v = 2t \sqrt{f} \). 2. The conservation of areas requires that \( v \propto (1+r) \). 3. \( r\sqrt{v}/(f+r)+v \). 4. \( r \propto \sqrt{fr} ; r' \propto \sqrt{fr'} \). 5. \( v' \propto \sqrt{(1-r)} \) \( (fr-f)^2/v = fr + v \). 6. \( 2fr + v \). 7. \( (2\pi \times \pi^2 + 4) = 2\pi v \); substituting for \( v \) its limit, \( 2fr + v \), we get \( \pi fr + v \).

8. \( t \) of planetary revolution at sun's surface \( = 1 \) year \( + \pi \sqrt{1214.86} = 10,020 \). \( v' = \sqrt{fr} = 2\pi + 10,020 ; v = 2\pi + (25.4 \times 86400) \). Therefore Prob. \( 5 = .13734r; 6 = .27468r; 7 = .4316r = v \) of light; for \( v \) of light \( = 214.86 \times 497.825 = .4316r \). Sun-spot observatons give rotation-periods varying between 24.6 days and 25.5 days. 9. \( v' = \sqrt{(3963 \times 5280 \times 32.08) + 5280 = 4.9m.; v = 24890 \div 86164 = .289m.; \pi fr + v = 261m. = v \) of planetary revolution at mean c. g. of Sun and Jupiter. Herschel's estimate for Jupiter's day is about 4 per cent, greater than that assumed in the problem.

10. Jupiter's dist. \( + \) by sum of masses \( = 5.2028 \times 214.86 + 1048.879 = 1.0658 ; \sqrt{(1.0658) \times 261 \times 10020 \times 214.86 + 2\pi = 92,115,000m. \)}

11. Estimating \( v \) of sound at .216m.; \( (185,034 \times 100,000,000 + .216) = 7,333,321,000,000,000,000,000,000,000,000,000. \) 12. It can have no inertia, and cannot, therefore, be a material medium.

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**Note, by Alex. Evans.**—I find that in the last edition, 1876, Sec. 514 of the Outlines of Astronomy by Sir J. F. W. Herschel, published by Appletons, N. Y., the period of Saturn's rotation is restored to \( 10^8 16^m 00.44^s \).
SOLUTIONS OF PROBLEMS IN NUMBER THREE.

167. "A solid sphere rolls down a trough formed by two planes which make with each other an angle $2\phi$. Find an expression for the time when the inclination of the trough to the horizon is $\theta$.”

SOLUTION BY PROF. C. M. WOODWARD, WASHINGTON UNIV., ST. LOUIS, MO.

The instantaneous axis of the body is the line $ab$ joining the two points of contact. $OC = R \sin \phi$, and since $OC$. $OC' = \frac{3}{2}R^2$ = the square of the radius of gyration, $C'$ is distant from the centre $OC = \frac{2R}{\sin \phi}$. Now the whole mass of the sphere may be supposed to be at $C$ and $C'$. $C$ has no motion, the mass at $C'$ moves freely under the influence of gravity as if sliding down a smooth inclined plane. Its acceleration is therefore $g \sin \theta$. The linear acceleration of the centre will therefore be

$$\frac{R \sin \phi \cdot g \sin \theta}{R \sin \phi + (2R + 3 \sin \phi)} = \frac{5 \sin^2 \phi}{5 \sin \phi + 2} \cdot g \sin \theta.$$ 

Since $s = \frac{1}{2} \cdot \frac{5g \sin^2 \phi \sin \theta}{5 \sin \phi + 2} \cdot t^2$, we have $t = \frac{1}{\sin \phi} \sqrt{\left[\frac{(10 \sin^2 \phi + 4)g}{5g \sin \theta}\right]}$.

168. "Find the general value of

$$u = \int \int \frac{dxdy}{\sqrt{(1 + x^2 + y^2)^3}}$$

and show that when the limits of $x$ and $y$ are 0 and 1, $u = 0.5$.”

SOLUTION BY PROF. H. T. EDDY, CINCINNATI UNIV., CIN., OHIO.

Transforming to polar coordinates,

$$u = \int_0^\pi \int_0^\zeta \frac{rdrd\theta}{\sqrt{(1 + r^2)^3}} + \int_0^\pi \int_0^\zeta \frac{rdrd\theta}{\sqrt{(1 + r^2)^3}}.$$ 

$\therefore u = \int_0^\zeta d\theta - \int_0^\zeta \frac{\cos \theta d\theta}{\sqrt{(1 + x^2 - \sin^2 \theta)}} + \int_0^\pi d\theta - \int_0^\pi \frac{\sin \theta d\theta}{\sqrt{(1 + y^2 - \cos^2 \theta)}}$,

where $\zeta = \tan^{-1}(y/x)$;

$\therefore u = \frac{\pi}{2} - \sin^{-1} \frac{y}{\sqrt{(x^2 + y^2)(1 + x^2)}} - \sin^{-1} \frac{x}{\sqrt{(x^2 + y^2)(1 + y^2)}}$. 

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\[ u = \frac{\pi}{2} - \tan^{-1} \left( \frac{\sqrt{1+x^2+y^2}}{xy} \right) = \tan^{-1} \left( \frac{ax}{\sqrt{1+x^2+y^2}} \right) \]

To obtain the general integral, add to the result \( \varphi(x) + \psi(y) \).

Between the limits 0 and 1 for both variables, \( \tan u = 1 + \sqrt{3} \); therefore

\[ u = \frac{\pi}{3} = 0.5 + \]

[Mr. Adcock gets for the general value

\[ \hat{u} = \frac{1}{4} \tan^{-1} \left[ \frac{(1+x^2+y^2)^{3/2} + y}{2x} - \frac{x^2 + 1}{x} \right] - \frac{1}{4} \tan^{-1} \left[ \frac{(1+x^2+y^2)^{3/2} + x}{2y} - \frac{y^2 + 1}{y} \right] - \frac{1}{4} \tan^{-1} \frac{1}{y} \]

---

169. "Base balls are covered by sewing together two dumb-bell shaped pieces of leather. Determine the shape of the pieces so as to reduce the distortion in fitting them to the spherical surface to a minimum."

SOLUTION BY CHAS. H. KUMMELL, U. S. LAKE SURVEY, DETROIT, MICH.

Conceive at first the covering to consist of three parts, viz., an equatorial belt and two caps at the poles. The belt is formed from a cylinder touching the ball at the equator, and, considering only one hemisphere, let \( p = \) height of this cylinder above plane of equator and \( r = \) radius of ball (which includes one half the thickness of the leather), then

\[ 2rp\pi = \text{surface of one half of equatorial belt.} \quad (1) \]

The caps at the poles are formed from a tangent plane at the poles; they are circles the radius of which let \( = q \), then

\[ q\pi = \text{surface of one cap at the pole.} \quad (2) \]

There are two extreme cases within which the correct solution of this problem must be comprised, viz:

1. The material is supposed to be absolutely inextensible.
2. The material is supposed extensible to allow being stretched over a curved surface of equal area to its plane area.

In the first case we must assume that the belt and polar cap, being bent over the ball, will meet, or

\[ p + q = \frac{1}{4} \pi r. \quad (3) \]

The distortion in this case is the excess of the covering areas over the area to be covered. Denoting this by \( T \) we have

\[ T = (2rp + q^2 - 2r^2)\pi = \text{minimum,} \quad (4) \]

or, introducing the condition (3), we have to make

\[ 2rp + q^2 - 2r^2 + (p + q - \frac{1}{4} \pi r)K = \text{min.,} \]

\( K \) being an undetermined constant.
We have then the two conditions for the minimum (denoting the minimum values of \( p \) and \( q \) by \( p_1 \) and \( q_1 \)),

\[
2r + K = 0, \quad 2q_1 + K = 0; \quad \therefore \quad q_1 = r. \quad (5)
\]

\[
\therefore \quad p_1 = \frac{1}{2}\pi r - r = 0.57080r. \quad (6)
\]

In the second case we have the condition

\[
2rpn + q^2 = 2r^2\pi \quad \text{or} \quad 2r^2p + q^2 = 2r^2. \quad (7)
\]

The distortion in this case is measured by the small belt left uncovered if the equatorial belt and polar caps are bent over the surface of the ball without being stretched. Call it \( t \); then if the arc \( s \) is reckoned from the eq'\( r \),

\[
t = 2\pi \int_0^s \cos r \cdot ds = 2\pi \left( \cos \frac{q}{r} - \sin \frac{p}{r} \right), \quad (8)
\]

or, introducing the condition (7),

\[
\cos \frac{q}{r} - \sin \frac{p}{r} + \frac{rp + \frac{1}{2}q^2 - r^2}{k^2} = \text{minimum},
\]

where \( k \) is an undetermined constant.

We have then the conditions for minimum (denoting the resulting values of \( p \) and \( q \) by \( p_2 \) and \( q_2 \))

\[
- \frac{1}{r} \sin \frac{q_2}{r} + \frac{q_2}{k^2} = 0, \quad - \frac{1}{r} \cos \frac{p_2}{r} + \frac{r}{k^2} = 0; \quad \therefore \quad \sin \frac{q_2}{r} = \frac{q_2}{r} \cos \frac{p_2}{r}. \quad (9)
\]

From (7) we have

\[
p_2 = 1 - \frac{q_2^2}{2r^2} \quad (10), \quad \therefore \quad \sin \frac{q_2}{r} = \frac{q_2}{r} \cos \left( 1 - \frac{q_2^2}{2r^2} \right). \quad (11)
\]

From these equations we find

\[
q_2 = 0.95388r,
\]

\[
p_2 = 0.54505r.
\]

The dumb-bell shaped pieces may be constructed as follows: Draw a parallelogram with the sides \( rr \) and \( 2q \) the altitude of which is \( 2p \). On the sides \( 2q \) describe semicircles with \( q \) as radius; the resulting figure, together with another piece of the same shape, is the best adapted for covering a ball. In the second and all intermediate cases between the first and second case, the circular edge which is to be sewed to the straight edge is slightly shorter; the stitches have therefore to be made closer on the circular edge and in order to secure the most perfect fit it might be necessary to mark the corresponding points beforehand.

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**QUERY** (proposed at p. 166). "How is the Rule, given at page 44, Gillespie's Land-surveying, (5th edition, New York, 1857,) demonstrated?"

[The rule referred to in the above query, as quoted by Dr. Oliver, is as follows:—"When the four sides and the sum of any two opposite angles are
given proceed thus: Take half of the sum of the four given sides, and from it subtract each side in turn. Multiply together the four remainders, and reserve the product. Multiply together the four sides. Take half their product and multiply it by the cosine of the given sum of the angles increased by unity. Regard the sign of the cosine. Subtract this product from the reserved product and take the square root of the remainder. It will be the area of the quadrilateral.”

**SOLUTION BY R. J. ADOCK, MONMOUTH, ILL.**

Let $a$, $b$, $c$, $d$ be the four sides of the quadrilateral $ABCD$, $a = AB$, $b = BC$, $c = CD$, $d = AD$, and $\frac{1}{2}(a+b+c+d) = h$, then it is required to show that

$$\text{area } ABCD = \sqrt{[(h-a)(h-b)(h-c)(h-d) - \frac{1}{2}abcd(\cos[A+C]+1)]}.$$  

Since $BD^2 = a^2 + d^2 + 2ad \cos A = b^2 + c^2 + 2bc \cos C,$  

$$2ad \cos A - 2bc \cos C = b^2 + c^2 - a^2 - d^2,$$

and $a^2d^2 \cos^2 A + b^2c^2 \cos^2 C - 2abcd \cos A \cos C = \frac{1}{4}(b^2+c^2-a^2-d^2)^2.$  

And since area $= \frac{1}{2}(ad \sin A + bc \sin C),$

$$a^2d^2 \sin^2 A + b^2c^2 \sin^2 C + 2abcd \sin A \sin C = 4(\text{area})^2.$$

Adding (2) and (3) and reducing,

$$a^2d^2 + b^2c^2 - 2abcd \cos (A + C) = 4(\text{area})^2 + \frac{1}{4}(b^2 + c^2 - a^2 - d^2)^2,$$

hence $(\text{area})^2 = \frac{1}{4}[4(a^2d^2 + b^2c^2) - (b^2 + c^2 - a^2 - d^2)^2] - \frac{1}{2}abcd \cos (A + C)$

$$= \frac{1}{4}[4a^2d^2 + 4b^2c^2 - (b^2 + c^2 - a^2 - d^2)^2 + 8abcd] - \frac{1}{2}abcd[1 + \cos(A+C)]$$

$$= (h-a)(h-b)(h-c)(h-d) - \frac{1}{2}abcd[1 + \cos(A+C)],$$

as required.

**SOLUTIONS OF PROBLEMS IN NUMBER FOUR.**

Solutions of problems in number 4 have been received as follows; From R. J. Adcock, 173 and answer to Mr. Baker’s query; Henry Gunder, 171; Henry Heaton, 170, 171 and 173; Prof. E. W. Hyde, 171; G. W. Hill, 170; Chas. H. Kummell, 170, 171 and 174; Artemas Martin, 172; E. B. Seitz, 170, 171 and .173.

170. “Given the lengths of the eight edges of a quadrangular pyramid to find its altitude.”
-153-

SOLUTION BY G. W. HILL, PH. D., NYACK, TURNPIKE, N. Y.

Denoting the solid angles by the symbols 0, 1, 2, 3, 4, of which 0 belongs to the vertex, and the edges by (01), (02), &c., let us divide the pyramid into two triangular pyramids by a plane passing through 0, 1, 3. Denote six time the volumes of the pyramids 0123 and 0134 severally by $A$ and $A'$, and twice the areas of their bases by $A^2$ and $A'^2$, and by $x$ their common altitude. Then we shall have

$$x^2 = \frac{A^2}{A^3} = \frac{A'^2}{A'^3},$$

whence

$$A'^2 A^2 = A^2 A'^2. \quad (1)$$

If we employ the notation

$$[12] = \frac{(01)+(02)-(12)}{2}, \quad [23] = \frac{(02)+(03)-(23)}{2}, \quad \&c.,$$

and, in order to distinguish it as an unknown quantity, put $y$ for $[13]$, we have for $A^2$, $A'^2$, $A^3$ and $A'^3$ the following expressions

$$A^2 = (01)(02)(03) - (01)[23]^2 - (03)[12]^2 + 2[12][23]y - (02)y^2,$$

$$A'^2 = (01)(03)(04) - (01)[34]^2 - (03)[14]^2 + 2[14][34]y - (04)y^2,$$

$$A^3 = \left[(01)+(02)-2[12]\right]\left[(02)+(03)-2[23]\right]-\left[(02)-[12]-[23]+y\right]^3,$$

$$A'^3 = \left[(03)+(04)-2[34]\right]\left[(01)+(04)-2[14]\right]-\left[(04)-[14]-[34]+y\right]^3.$$

For the proof of these equations see a memoir of Lagrange, Solutions analytiques de quelques problèmes sur les Pyramides Triangulaires. Tome III, p. 659.

On the substitution of these values in (1), we have an equation of the fourth degree in $y$, which serves to determine this quantity and thence $x$.

171. "A pail in the form of a frustrum of a cone—radius of upper base 2$r$, of lower base $r$, and height $h$—is inclined so that if water be poured in to it, the water will just reach the lower edge of the upper base, and the upper edge of the lower base. How many gallons of water in the pail?"

SOLUTION BY PROF. E. W. HYDE, UNIV. OF CINCINNATI, CIN., OHIO.

Let $ABCD$ (see fig. on next page) be the pail and $AC$ the surface of the water, then we have to find the volume $ABC = AVC - AVB$. Volume $AVC = \text{(area of ellipse } AC) \times \frac{1}{2} p$, in which $p = VE = \text{perp. from } V \text{ on } AC$. 

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Volume of \(\Delta VBC = \frac{1}{3} h \times \pi r^2 = \frac{1}{3} \pi r^2 h.\)

From the figure \(a = \sin^{-1} \frac{h}{(h^2 + 9r^2)^{\frac{1}{2}}}\), \(\beta = \sin^{-1} \frac{3r}{(h^2 + r^2)^{\frac{1}{2}}}\), \(\gamma = \beta - a.\)

\[ p = VCSin \gamma = 2\sqrt{(h^2 + r^2)} \sin(\beta + a) = 2\sqrt{(h^2 + r^2)} \left[ \frac{3hr - hr}{\sqrt{(h^2 + 9r^2)} \sqrt{(h^2 + r^2)}} \right] \]

\[ = \frac{4hr}{\sqrt{(h^2 + 9r^2)}}.\]

Let \(a\) and \(b\) be the semi axes of the ellipse \(AC\); then its area is \(\pi ab.\)

\[ a = \frac{1}{4} AC = \frac{1}{4} \sqrt{(h^2 + 9r^2)}, \quad b = FG = \sqrt{[(\frac{3r}{2})^2 - (\frac{r}{2})]} = r/2; \quad \pi ab = \frac{1}{4} \pi \sqrt{2} \times \sqrt{(h^2 + 9r^2)}.\]

\[ \therefore \text{Vol. } AVC = \frac{4hr}{3\sqrt{(h^2 + 9r^2)}} \cdot \frac{1}{2} \pi \sqrt{\frac{2}{\sqrt{(h^2 + 9r^2)}}} = \frac{1}{2} \pi h r^3.\]

Hence \(\text{Vol. } ABC = \frac{1}{3} \pi h r^3 - \frac{1}{2} \pi h r^3 = \frac{1}{2}(\sqrt{2} - 1) \pi h r^3.\)

172. "Find the least integral values of \(x\) and \(y\) that will satisfy the equation \(x^2 - 9817y^2 = 1.\"

**SOLUTION BY ARTEMAS MARTIN, M. A., ERIE, PA.**

Put \(A = 9817\), then \(\sqrt{A} = \sqrt{(9817)} = r + \frac{1}{u_1 + \frac{1}{u_2 + \frac{1}{u_3 + \text{ etc.}}}}\)

where \(r\) is the integral part of \(\sqrt{A}.\)

The last quotient of every complete period is \(2r.\) Let \(m\) be the number of quotients in a complete period, and \(p_m \div q_m\) the last convergent in the first period; then, when \(m\) is even, \(x = p_m, y = q_m\), and when \(m\) is odd, \(x = p_{2m}, y = q_{2m}.\)

Let \((r + a_n) \div b_n = u_n + \text{ etc. and } (r + a_{n+1}) \div b_{n+1} = u_{n+1} + \text{ etc. be any two consecutive complete quotients, then } a_0 = 0, b_0 = 1; a_1 = r, b_1 = A - r^2; u_0 = r, u_1 = 2r \div (A - r^2); a_{n+1} = u_n b_n - a_n, b_{n+1} = (A - a_{n+1}) \div b_n.\)

If \(p_n + q_n, p_{n+1} + q_{n+1}\) be any two consecutive convergents and \(u_{n+1}\) the quotient corresponding to \(p_{n+1} + q_{n+1},\) then
\[ p_1 = r \quad p_2 = ru_1 + \frac{1}{2} \quad \ldots \quad p_{n+2} = \frac{u_{n+1} p_{n+1} + p_n}{u_{n+2}}. \]

The partial quotients are easily found to be

\[ q_1 = 1, \quad q_2 = 2, \quad \ldots, \quad q_{n+2} = \frac{u_{n+1} q_{n+1} + q_n}{u_{n+2}}. \]

As 95, the number of quotients in a period, is odd, therefore \( x = p_{95}, y = q_{95} \) satisfy the equation \( x^2 - 9817y^2 = -1 \); and \( x = p_{100}, y = q_{100} \) satisfy the equation \( x^2 - 9817y^2 = +1 \).

It is not necessary to compute the numerators of the convergent fractions as \( p_n = rq_n + q_{n-1} \). Computing the values of \( q_1, q_2, q_3, \ldots \), we find

\[ q_{95} = 744173479920444607112253030323151170560356045, \]
\[ p_{95} = 73733285220349094575924116358509678588651469062. \]

\[ x = p_{100} = 2p_{95} + 1 = 108731946987700456541715000199726898 \]
\[ 78078955845585116794522041943260442884680816719737118849, \]
\[ y = q_{100} = 2p_{95}q_{95} = 10974071089678774410161078963233070156 \]
\[ 422894010351506814076536718633072745503799243013892130880. \]

These numbers are believed to be the largest of the kind that have yet been found.

173. "A Given semicircle is divided into two quadrants, and a point is taken at random in each quadrant; find the chance that the distance between them is less than the radius of the semicircle."

**Solution by E. B. Seitz, Greenville, Ohio.**

Let \( AOD \) and \( BOD \) be the two quadrants, \( P, Q \) the random points. With \( D, P \) as centers and radii each equal to the radius of the semicircle, describe the arcs \( OC, EF \). Join \( OP, OF, PF, PE \).

If \( P \) is in the surface \( OCD \), and \( Q \) in the surface \( OEFD \), the distance between them is less than the radius of the semicircle.

Put \( OA = 1, OP = x, \angle POD = \theta, \) and area \( OEFD = u \). Then

\[ u = \frac{1}{2} \sin^{-1}(x \cos \theta) + \cos^{-1}(\frac{1}{2}x) - \frac{1}{2}x\sqrt{(4-x^2)} - \frac{1}{2}x \cos \theta \sqrt{(1-x^2 \cos^2 \theta)} - \frac{1}{2}x^2 \sin \theta \cos \theta. \]

But when \( P \) is in the surface \( OAC \), the arc described from it does not cut the arc \( BD \), and the area cut off from the quadrant \( BOD \) is
\[ u_1 = \frac{1}{2} \sin^{-1}(x \cos \theta) + \frac{1}{2} \cos^{-1}(x \sin \theta) + \frac{1}{2} x \cos \theta \sqrt{1-x^2 \cos^2 \theta} \]

\[ - \frac{1}{2} x \sin \theta \sqrt{1-x^2 \sin^2 \theta} - x^3 \sin \theta \cos \theta. \]

Hence, since the whole number of ways the two points can be taken is \( \frac{1}{2} \pi^2 \), we have for the required chance

\[
p = \frac{16}{\pi^2} \int_0^{\pi} \int_0^{\frac{1}{2} \pi} u(x \cos \theta + 2 \sec \theta + 2 \tan \theta - 6 \sqrt{3}) d\theta
+ \frac{1}{2 \pi} \int_0^{\pi} \int_0^{\frac{1}{2} \pi} (16 \pi - 32 \theta - 2 \pi \sec \theta - 2 \pi \sec \theta + 2 \tan \theta + 6 \theta \sec \theta - 6 \cot \theta - 8 \sin \theta \cos \theta + 128 \sin \theta \cos \theta - 64 \sin \theta \cos \theta + 64 \sin \theta \cos \theta) d\theta
\]

\[= \frac{4}{3} - \frac{\sqrt{3}}{\pi} - \frac{2}{\pi^2}. \]

[Mr. Heaton, by a similar process, gets \( p = \frac{4}{3} - \frac{\sqrt{3}}{\pi} - \frac{1}{\pi^2} \); and Mr. Adcock, by restricting the random points to the periphery of the quadrants, gets \( p = \frac{4}{3} \). We have not verified the operations in either Mr. Seitz' or Mr. Heaton's solution, and therefore cannot say which is correct.]

174. "Give the most convenient method to compute \( \Gamma \left( \frac{1}{n} \right) \), \( n \) being an integer."

SOLUTION BY CHAS. H. KUMMELL, DETROIT, MICH.

Because \( \Gamma \left( \frac{1}{n} \right) = \left( \frac{1}{n} - 1 \right) \Gamma \left( \frac{1}{n} - 1 \right) = (\frac{1}{n} - 1)(\frac{1}{n} - 2) \ldots (\frac{1}{n} - r) \Gamma \left( \frac{1}{n} - r \right) \), (1)

where \( 1 > \frac{1}{n} - r > 0 \), it is only necessary to determine

\( \Gamma \left( \frac{1}{n} \right), \Gamma \left( \frac{1}{n} - 1 \right) \).

Of these \( \Gamma \left( \frac{1}{n} \right) = \sqrt{\frac{\pi}{n}} \) is known.

By the theorem:

\[ \Gamma \left( n - 1 \right) = \pi \csc \pi n \]

we have

\[ \Gamma \left( \frac{1}{n} \right) \Gamma \left( \frac{1}{n} - 1 \right) = \pi \csc \pi \frac{1}{n} = 2 \pi, \ldots \Gamma \left( \frac{1}{n} \right) = 2 \pi + \Gamma \left( \frac{1}{n} \right), \]

\[ \Gamma \left( \frac{1}{n} - 1 \right) \Gamma \left( \frac{1}{n} - 2 \right) = 2 \pi - \pi \csc \pi \frac{1}{n}, \ldots \Gamma \left( \frac{1}{n} \right) = 2 \pi - \pi \csc \pi \frac{1}{n} + \Gamma \left( \frac{1}{n} \right). \]

It is then only necessary to compute \( \Gamma \left( \frac{1}{n} \right) \) and \( \Gamma \left( \frac{1}{n} \right) \). By Gauss' theorem:

\[ \Gamma \left( n + \frac{1}{r} \right) \Gamma \left( n + \frac{2}{r} \right) \ldots \Gamma \left( n + \frac{r-1}{r} \right) = (2 \pi)^{\frac{r-1}{2}} r^{-m+\pi} \Gamma \left( \frac{n}{r} \right), \]

we have, placing \( n = \frac{1}{r} \) and \( r = 2 \), \( \Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{3}{2} \right) = 2^{n-3} \Gamma \left( \frac{3}{2} \right) \Gamma \left( \frac{3}{2} \right) \); hence by (4)

\[ \Gamma \left( \frac{1}{2} \right) = 2^{-\pi} 3^{\frac{1}{2}} \pi^{-\frac{1}{2}} \Gamma \left( \frac{3}{2} \right)^2. \]

If then \( \Gamma \left( \frac{1}{2} \right) \) is known \( \Gamma \left( \frac{1}{n} \right) \) is known.
To compute this we employ the definite integral

$$u = \int_0^{2\pi} \frac{d\phi}{(\sin \phi)^3} = \frac{3}{2} B(\frac{1}{3}, \frac{1}{3}) = \frac{\Gamma(\frac{1}{3}) \Gamma(\frac{2}{3})}{2\Gamma(\frac{1}{3})}$$

$$= 2^{-3} 3^{3\pi - \pi} \Gamma(\frac{1}{3}) \Gamma(\frac{2}{3}) \text{ by (4), } = 2^{-1} 3\pi - 1 \Gamma(\frac{1}{3}) \text{ by (6).}$$

Therefore

$$\Gamma(\frac{1}{3}) = 2^{-1} 3(\pi x)^{\frac{1}{3}}. \text{ (7)}$$

The integral $u$ may be evaluated as follows:

Place

$$\sin \phi = \left(1 + \sqrt{3} \cot \frac{\phi}{2}\right)^{-1},$$

$$\therefore \cos \phi d\phi = \frac{3}{2} \cot \frac{\phi}{2} d\phi = \frac{3}{2} \cot \frac{\phi}{2} (\sin \phi)^{\frac{1}{2}}$$

$$\cos \phi = \sin \phi \sqrt{[3 \sqrt{3} (\cot \frac{\phi}{2} + \sqrt{3} \cot \frac{\phi}{2} + \cot \frac{\phi}{2})]}$$

$$= 3^{3} \sin \phi \cot \frac{\phi}{2} \sqrt{[(1 - \cos \phi)^2 + \sqrt{3} \sin \phi + (1 + \cos \phi)^2]}$$

$$= 2^{-1} 3^{3} \sin \phi \cot \frac{\phi}{2} \sqrt{[2 \sin \phi + (\sqrt{3} + \sqrt{3}) \sin \phi]};$$

therefore, since $\phi = 0$ if $\phi = 0$, and $\phi = \pi$ if $\phi = \pi$,

$$u = \int_0^{2\pi} \frac{d\phi}{(\sin \phi)^3} = \int_0^{\pi} \frac{3^{3} d\phi}{\sqrt{[2 \sin \phi + (\sqrt{3} + \sqrt{3}) \sin \phi]}}. \text{ (8)}$$

Comparing this with form (9) ANALYST, Vol. IV, page 121, we have

$$a_0 = 2; b_0 = \sqrt{\frac{1}{3} + \sqrt{\frac{3}{}}}; \text{}$$

$$= 1.93185165; \psi_0 = \pi.$$}

The arithmetic-geometric mean denoted there by $b_0$ might be better denoted thus

arithmetic-geometric mean of $a_0$ and $b_0$ $= \frac{1}{2} \left\{ a_0 + b_0 \right\}$$

In this case we have

$$\frac{1}{2} \left\{ 2^+ \left( \sqrt{\frac{1}{3} + \sqrt{\frac{3}{}}} \right) \right\}^{\frac{1}{2}} = 1.96577817,$$

and

$$\psi_1 = 2\pi, \psi_2 = 2\pi, \ldots \psi_* = 2^{\ast} \pi.$$

Applying formula (13), ibid., we have

$$u = \frac{3^{3} \psi_*}{2^{*} \left\{ a_0 + b_0 \right\}^{\frac{1}{2}}} = \frac{3^{3} \pi}{2^{*} \left\{ 2^+ \left( \sqrt{\frac{1}{3} + \sqrt{\frac{3}{}}} \right) \right\}^{\frac{1}{2}}}.$$

We have then by (7)

$$\Gamma(\frac{1}{3}) = 2^{-1} 3^{3\pi^{\frac{1}{2}}} \left\{ 2^+ \left( \sqrt{\frac{1}{3} + \sqrt{\frac{3}{}}} \right) \right\}^{\frac{1}{2}}.$$

This together with (1) (3) (4) and (6) gives $\Gamma(\frac{1}{3} n)$. 

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QUERY. "In the Treatise on the elements of Geometry by Bouche' and Comberousse, some miscellaneous problems in solid geometry are proposed, among which is the following:—'Draw a sphere which shall cut five other spheres under equal angles.'—Can any reader of the Analyst refer me to a solution, or if not, give one?"

ANSWER BY R. J. ADCOCK.

Let $r_1$, $r_2$, $r_3$, $r_4$, $r_5$ be the radii of the given spheres, $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, &c., the coordinates of their centres, $\delta$ the radius of the required sphere and $\alpha$, $\beta$, $\gamma$, the coordinates of its centre. And put $A$ = the angle of intersection. Then the square of the line joining the centres of the spheres $r_1$, and $\delta$ is

$$\pm \cos A = \frac{(x_1-\alpha)^2 + (y_1-\beta)^2 + (z_1-\gamma)^2 - r_1^2 - \delta^2}{2r_1 \delta}$$

$$= \frac{(x_2-\alpha)^2 + (y_2-\beta)^2 + (z_2-\gamma)^2 - r_2^2 - \delta^2}{2r_2 \delta}$$

$$= \frac{(x_3-\alpha)^2 + (y_3-\beta)^2 + (z_3-\gamma)^2 - r_3^2 - \delta^2}{2r_3 \delta}$$

$$= \frac{(x_4-\alpha)^2 + (y_4-\beta)^2 + (z_4-\gamma)^2 - r_4^2 - \delta^2}{2r_4 \delta}$$

$$= \frac{(x_5-\alpha)^2 + (y_5-\beta)^2 + (z_5-\gamma)^2 - r_5^2 - \delta^2}{2r_5 \delta},$$

four equations for the four quantities $\alpha$, $\beta$, $\gamma$, $\delta$, which completely determine the required sphere.

PROBLEMS.

175. **By Newton Fitz, Norfolk, Virginia.**—Find the roots of the equation

$$x^4 + Ax^3 + Bx^2 + Cx + \frac{C^2}{A^3} = 0.$$  

176. **By Prof. Obson Pratt, Sen., Salt Lake City, Utah.**—Given

$$u = F(y), \text{ and } y = F'[x + xyf(y)];$$

also

$$v = F_1(t), \text{ and } t = F_2[x + xyf_1(t)],$$

to expand (by the differential calculus) $u$ in a series of ascending, positive and integral powers of $x$. ($x$ not being a function of $x$.)
177. By Christine Ladd, Union Springs, N. Y.—If \( I_1, I_2, I_3 \) be the points of contact of the inscribed circle with the sides of the triangle \( ABC \), \( E_1, E_2, E_3 \) the centres of the escribed circles, \( r \) the radius of the circle inscribed in \( I_1I_2I_3 \) and \( r' \) the radius of that inscribed in \( E_1E_2E_3 \), show that

\[
    r = r' \frac{a + b + c}{a + \beta + \gamma} \quad r' = 2R \frac{a + b + c}{a + \beta + \gamma}
\]

where \( R \) and \( r \) have their usual values and \( a, \beta, \gamma \) are the distances between the centres of the escribed circles.

178. By Marcus Baker, U. S. Coast Sur., Wash., D. C.—Through any point \( O \) in a plane triangle \( ABC \) three lines \( a, \beta, \gamma \) are drawn parallel to the sides \( a, b \) and \( c \) respectively; prove that

\[
    \frac{a}{a} + \frac{\beta}{b} + \frac{\gamma}{c} = 2.
\]


\[
    \frac{\sin \theta d\theta}{(\sin \theta + \cos \theta)^3}.
\]

180. (Selected) By Geo. Lilley, Kewanee, Ill. — Two equal spheres placed in a paraboloid with its axis vertical touch one another at the focus. If \( W \) be the weight of a sphere, \( R, R' \) the pressures upon it prove that

\[
    W^2 : R : R' :: 3 : 2.
\]

181. By Prof. E. W. Hyde.—Find the volume between \( z = 0 \) and \( z = 2l \) of the solid bounded by the surface whose equation is

\[
    a(y^2 + z^2) - z^2(x^2 - 2ay^2 + 2x^2) - y^2(bx^2 + c^2x + c^2) = 0.
\]

**Special Notice.**—As is well known to the readers of the *Analyst*, its publication was undertaken, not with the expectation of gain, but to supply, temporarily, a "medium of communication" between mathematicians and students of mathematics in this country; and it has been, and is, our intention only to continue the publication so long as it will be of service to mathematicians.

Our patronage, so far, has exceeded our anticipations, so that we had no thought of discontinuing the publication at present. But, through the kindness of Prof. Hall, we have just received (Aug. 11th) a Prospectus of *The American Journal of Pure and Applied Mathematics*, to be published at Baltimore; and as most of our subscribers will undoubtedly want the new Journal, and many of them may not want to incur the expense of two
mathematical journals, we copy, below, the prospectus alluded to, and ask, as a special favor, that all who may desire to discontinue their subscriptions for the Analyst at the close of Vol. IV, will notify us before the 1st of October, so that we may be able to announce our intentions for the future in No. 6 of the Analyst.

Editor of Analyst.

"The American Journal of Pure and Applied Mathematics will have for its primary object to afford a medium of intercourse between original investigators. In this work, although the editors are convinced that they must look to the mathematicians of America for their principal support, they confidently anticipate valuable contributions from European mathematicians of eminence.

"For the benefit of teachers and students of mathematics there will appear from time to time brief treatises on such modern and higher subjects as may seem useful or interesting.

"The editors propose also to review the most important mathematical publications that may appear in this country and in Europe, thus keeping their subscribers informed as to the best articles and treatises written on the various mathematical subjects.

"The quarto form of page has been selected as best adapted to mathematical papers.

"The journal will be published in numbers appearing, for the present, four times a year, and 384 pages will constitute a volume, for which subscriptions will be received. Price Five Dollars a volume, payable in advance. The first number will appear in January, 1878.

"Subscriptions and contributions may be addressed to

WILLIAM E. STORY,
Johns Hopkins University, Baltimore, Maryland."

ERRATA.

On page 99, last line, for \( \log K + \log (1 + aP) = \beta \log (PV) \),
read \( \log K + \log (1 + aP)^\beta = \log (PV) \).

" 123, line 6, from bottom, for A. W. Whitaker, read Dr. A. W. Whitcom.
" 131, line 11, for \( \sin^2 \theta \), in denominator, read \( \sin \theta \).
" 141, line 3, for \( BC \), read Bc.
" " lines 16 and 17, from bottom, for \( X \bar{z} = -Y \bar{y} \), read \( X' \bar{z} = -Y' \bar{y} \).
" " 142, line 2, dele "all".
" " 149, line 8, for \( O' \), read \( O' \) the centre of percussion.
SHORT METHOD OF ELLIPTIC FUNCTIONS.

BY LEVI W. MEECH, A. M., HARTFORD, CONN.

(Continued from page 136)

When the last modulus is so small that it may be taken as 0, the denominator, \(\sqrt{1-e^2 \sin^2 \theta_a}\), evidently reduces to its first term 1; whence \(F(0, \theta_a) = \theta_a\); where \(\theta_a\) denotes the corresponding limit-amplitude. Then,

\[
F(\epsilon, \theta) = \theta_a \frac{1}{2} (1+\epsilon^2) \frac{1}{2} (1+\epsilon^0) \frac{1}{2} (1+\epsilon^0) \ldots \frac{1}{2}.
\]

Let

\[
A = (1+\epsilon^2)(1+\epsilon^0)(1+\epsilon^0) \ldots = \frac{2}{1+b} \cdot \frac{2}{1+b^0} \cdot \frac{2}{1+b^0} \ldots = \frac{\sqrt{b^0 b^0 b^0 \ldots}}{b}.
\]

Also let \(n\) denote the number of approximations, or of factors in the numerator; then

\[
F(\epsilon, \theta) = A.(\theta_a + 2^n) = A\pi.
\]

When \(\theta = \frac{1}{4}\pi\), the formula of amplitude gives \(\theta' = \pi\); \(\theta'' = 2\pi\); \(\theta''' = 4\pi\); etc. Hence, for the quadrantal value,

\[
F(\epsilon, \frac{1}{4}\pi) = A\cdot\frac{1}{4}\pi.
\]

To aid in judging of the approximation, we take \(\epsilon = \sin 22^{1/2}\), or \(b = \cos 22^{1/2}\), then

\[
e^0 = \sin 2^{16}16'03" = .03956, \quad b^0 = \cos 2^{16}16'03" = .99922,
\]

\[
e^0 = \sin 0^{13}28" = .00039, \quad b^0 = \cos 0^{13}28" = .99999991,
\]

\[
e^0 = .000 000 038 35+ \quad b^0 = 999 999 999 999 999 26.
\]

Again, let \(\epsilon = \sin 45^\circ\), or \(\epsilon^2 = \frac{1}{2} = 0.5\); then in common logarithms,

\[
\log \epsilon, \sin 45^\circ = 9.84948500 \quad \log b, \cos 45^\circ = 9.84948500
\]

\[
\log \epsilon, \sin 9^\circ52'45" = 9.23444863 \quad \log b, \cos 9^\circ07'15" = 9.99351181
\]

\[
\log \epsilon, \sin 9^\circ25'40" = 7.87330122 \quad \log b, \cos 8^\circ34'20" = 9.99998788
\]

\[
\log \epsilon, \sin 9^\circ25'40" = 5.14455457 \quad \log b, \cos 8^\circ59'57" = 9.99999999
\]

\[
\frac{1}{2}(\epsilon^2)^2 = .000 000 000 048 65 \quad A = 0.072 007 344 8.
\]

\[
e^0 = 2^0 - (1-b^0) \sin 2^0 + \frac{1}{2} (1-b^0)^2 \sin 4^0 - \ldots
\]
This last series is derived from the formula of amplitudes by the well-known trigonometric development. When \( \theta \) is less than 45°, the last written term can evidently be omitted, and the result will still be exact to ten decimal places; that is \((1 - b^{\infty})/(1 + b^{\infty}) = \frac{1}{4}e^{\tan^2} = \frac{1}{4}\tan^2\varphi^o\). And by reduction, \( A = \left(\frac{2}{1+b}\right)^{\frac{1}{4}} \cdot \frac{2}{1+\sqrt{b}} = \sqrt[4]{(\cos \varphi + 1) \cdot \sqrt{(\cos \frac{1}{2} \varphi)}}\).

\( F(e, \theta) = A \cdot \left(\frac{1}{4}e^{\tan^2} \cdot \sin 2\theta^o\right) = A \cdot \left(\frac{1}{4}e^{\tan^2} \cdot \sin 4\theta^o\right) \)

When \( e^2 \) is greater than \( \frac{1}{2} \), we may use the formulas of ascending moduli, Sect. IV, and the third approximate value or \( \theta'' \) will still be exact to ten decimals.

**Note 1.** Another new scale of amplitudes will be developed in Section VII.

**Note 2.** By shorter process from Sect. VIII, let \( \mathcal{F} = \theta + 2u + v; \tan u = \sqrt{b} \tan \theta; \tan v = b \tan \theta; \) then

\[
\frac{d\theta}{\sqrt{(1 - e^2 \sin^2 \theta)}} = \frac{1}{1 + b} \cdot \frac{1}{1 + b} \cdot \sqrt{(1 - e^2 \sin^2 \mathcal{F})} \frac{d\mathcal{F}}{1 + 2 \mathcal{F}}
\]

**Note 3.** Again, as derived from Jacobi, let the common logarithm of \( q \), that is, \( \log q = 0.4332944819 \times \pi A' + A \); where \( A' + A = F(b, \frac{1}{2} \pi) + F(e, \frac{1}{2} \pi) \); also \( \log 0.433294482 \pi = 0.1349341840 \); and by Sect. VII,

\[
A = \sqrt{b} \cdot \frac{1 - q - q^4 + 2q^4 - q^8 - \ldots}{1 + q - q^4 - 2q^4 - q^8 - \ldots}; \tan \frac{\mathcal{F} - \theta}{2} = A \tan \theta; e^2 = \frac{e^2}{(1 + \mathcal{J})^2};
\]

then

\[
\frac{d\theta}{\sqrt{(1 - e^2 \sin^2 \theta)}} = \frac{1}{1 + 2 \mathcal{J}} \cdot \frac{d\mathcal{F}}{\sqrt{(1 - e^2 \sin^2 \mathcal{F})}}.
\]

V. **Integration by Series.** As shown by analytical trigonometry, if \( e = \sin \varphi \), then \( dF \) or

\[
\frac{d\theta}{\sqrt{(1 - e^2 \sin^2 \theta)}} = \frac{1}{2 \cos^2 \varphi \sqrt{(1 + \tan^2 \frac{1}{2} \varphi \cdot e^{2\theta} - 1)(1 + \tan^2 \frac{1}{2} \varphi \cdot e^{-2\theta} - 1)}}
\]

Expanding and integrating

\[
F = \frac{1}{\cos^2 \varphi} \left\{ \theta \left[ 1 + \left( \frac{1}{2} \tan \varphi \right)^2 + \left( \frac{3}{8} \tan \varphi \right)^2 + \left( \frac{5}{16} \tan \varphi \right)^2 + \ldots \right] 
- \sin 2\theta \left[ \frac{1}{2} \tan \varphi \frac{1}{2} + \frac{3}{16} \tan \varphi \frac{1}{2} + \frac{15}{128} \tan \varphi \frac{1}{2} + \ldots \right] 
+ \sin 4\theta \left[ \frac{3}{8} \tan \varphi \frac{3}{8} + \frac{5}{32} \tan \varphi \frac{5}{32} + \ldots \right] 
- \sin 6\theta \left[ \frac{5}{16} \tan \varphi \frac{5}{16} + \ldots \right] + \text{etc.} \right\}.
\]

In like manner, for the function \( E \), of the second species,

\[
dE = \cos^2 \frac{1}{2} \varphi \sqrt{(1 + \tan^2 \frac{1}{2} \varphi \cdot e^{2\theta} - 1)(1 + \tan^2 \frac{1}{2} \varphi \cdot e^{-2\theta} - 1)} d\theta.
\]
Expanding and integrating as before,

\[ E = \cos \frac{1}{2} \varphi \left\{ 1 + (\frac{1}{2}) \tan \frac{1}{2} \varphi + (\frac{1}{8}) \tan^3 \frac{1}{2} \varphi + (\frac{1}{16}) \tan^5 \frac{1}{2} \varphi + \ldots \right\} \\
+ \sin 2\varphi \left[ \frac{1}{2} \tan \frac{1}{2} \varphi + \frac{1}{8} \tan^3 \frac{1}{2} \varphi + \frac{1}{16} \tan^5 \frac{1}{2} \varphi + \ldots \right] \\
\frac{1}{8} \sin 4\varphi \left[ \frac{1}{2} \tan \frac{1}{2} \varphi + \frac{1}{8} \tan^3 \frac{1}{2} \varphi + \ldots \right] + \text{etc.} \}

When \( \theta = \frac{1}{2} \pi \) the preceding series for \( F \) and \( E \) reduce to their first terms. Having computed these, the remaining terms can be accurately derived from each other by the following process from Legendre, ("Fonctions Elliptiques") Vol. 1, p. 276.

\[ F = \int \frac{d\vartheta}{\sqrt{1 - \epsilon^2 \sin^2 \theta}} = A_0 - A_1 \sin 2\vartheta + A_2 \sin 4\vartheta - A_3 \sin 6\theta + \ldots \]

Differentiating once, and again,

\[ \frac{dF}{d\vartheta} = \frac{1}{\sqrt{1 - \epsilon^2 \sin^2 \theta}} = A - 2A_1 \cos 2\vartheta + 4A_2 \cos 4\vartheta - 6A_3 \cos 6\theta + \ldots \]

\[ \frac{\epsilon^2 \sin 2\vartheta}{2\sqrt{1 - \epsilon^2 \sin^2 \theta}} = 4A_1 \sin 2\vartheta - 16A_2 \sin 4\vartheta + 36A_3 \sin 6\theta + \ldots \]

Multiplying the last equation by \( 2(1 - \epsilon^2 \sin^2 \theta) \div \epsilon^2 \), or by \( (2\div \epsilon)^{-1} - 1 + \cos 2\vartheta \), to render the left hand members identical, we equate the coefficients of \( \sin 2\vartheta, \sin 4\theta, \ldots \) in the right hand members; thus, we shall find,

\[ 2 \times 3A_2 = 4A_1 [(2 \div \epsilon^2) - 1] - A, \]
\[ 3 \times 5A_3 = 16A_2 [(2 \div \epsilon^2) - 1] - 1 \times 3A_1, \]
\[ 4 \times 7A_4 = 36A_3 [(2 \div \epsilon^2) - 1] - 2 \times 5A_2, \]
\[ 5 \times 9A_5 = 64A_4 [(2 \div \epsilon^2) - 1] - 3 \times 7A_3, \ldots \]

Again assuming for \( E \) and differentiating, we have,

\[ E = \int d\vartheta \frac{\sqrt{1 - \epsilon^2 \sin^2 \theta}}{\sqrt{1 - \epsilon^2 \sin^2 \theta}} = B_0 + B_1 \sin 2\vartheta - B_2 \sin 4\vartheta - B_3 \sin 6\theta - \ldots \]

\[ \frac{dE}{d\vartheta} = \sqrt{1 - \epsilon^2 \sin^2 \theta} = B + 2B_1 \cos 2\vartheta - 4B_2 \cos 4\vartheta + 6B_3 \cos 6\theta - \ldots \]

\[ \frac{\frac{1}{8} \epsilon^2 \sin 2\vartheta}{\sqrt{1 - \epsilon^2 \sin^2 \theta}} = 4B_1 \sin 2\vartheta - 16B_2 \sin 4\vartheta + 36B_3 \sin 6\theta + \ldots \]

Multiplying the preceding derivative of \( F \) by \( \frac{1}{8} \epsilon^2 \sin 2\vartheta \), to make the left hand member identical with this last equation; than equating the coefficients of \( \sin 2\vartheta, \sin 4\theta, \ldots \) in the right hand members,

\[ 4B_1 = \frac{1}{8} \epsilon^2 (A - 2A_2), \]
\[ 16B_2 = \frac{1}{8} \epsilon^2 (A_1 - 3A_3), \]
\[ 36B_3 = \frac{1}{8} \epsilon^2 (2A_2 - 4A_4), \ldots \]
\[ 4n^2 B_n = \frac{1}{8} \epsilon^2 [(n - 1)A_{n-1} - (n + 1)A_{n+1}]. \]
### Table of Elliptic Quadrants of the First and the Second Species.

Also the Common Logarithm of \( q \). Here \( e = \sin \varphi \).

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<th>( E(e, \frac{1}{2}\pi) )</th>
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**Note.** The 10, arbitrarily added, is to be subtracted from Log $q$; thus for 45°, Log $q$ is really 2.63563. And for 90°, Log $q$ is 0.
The equation for $A_1$ is easily found by substituting the series for $F$ and for $E$ in the identical equation,

$$dE = A d\theta = 4d\theta + A = (1 - \frac{1}{4}e^2 + \frac{1}{4}e^2 \cos 2\theta) dF.$$  

Whence $B = \left(1 - \frac{\theta^2}{2}\right) A - \frac{\theta^2}{2} A_1$, or $A_1 = A \left( \frac{2}{\theta^2} - 1 \right) - \frac{2}{\theta^2} B$.

Had we developed directly by the binomial theorem, and then integrated, we should have found, for quadrants,

$$F(e, \frac{1}{2} \pi) = A \frac{1}{2} \pi = \frac{\pi}{2} \left[ 1 + \left( \frac{1}{2} \right)^2 e^2 + \left( \frac{1.3}{2.4} \right) e^3 + \left( \frac{1.3.5}{2.4.6} \right) e^4 + \ldots \right].$$

$$E(e, \frac{1}{2} \pi) = B \frac{1}{2} \pi = \frac{\pi}{2} \left[ 1 - \left( \frac{1}{2} \right)^2 e^2 + \left( \frac{1.3}{2.4} \right) e^3 + \left( \frac{1.3.5}{2.4.6} \right) e^4 + \ldots \right].$$

Here

$$A = -e^2 \frac{d}{de} \left( \frac{B}{e} \right).$$

The exponential developments have just given, with $e = \sin \varphi$,

$$A = \frac{1}{\cos ^{3} \frac{1}{2} \varphi} \left[ 1 + \frac{1}{2} \tan ^{4} \frac{1}{2} \varphi + \left( \frac{\varphi}{2} \right)^{2} \tan ^{4} \frac{1}{4} \varphi + \left( \frac{\varphi}{2} \right)^{3} \tan ^{12} \frac{1}{12} \varphi + \ldots \right],$$

$$B = \cos ^{3} \frac{1}{2} \varphi \left[ 1 + \frac{1}{2} \tan ^{4} \frac{1}{2} \varphi + \left( \frac{\varphi}{2} \right)^{2} \tan ^{4} \frac{1}{4} \varphi + \left( \frac{\varphi}{2} \right)^{3} \tan ^{12} \frac{1}{12} \varphi + \ldots \right].$$

Lastly, to develop after a single decrease of modulus, let $t$ be determined by the relation $\sqrt{b} = \sqrt{(\cos \varphi)} = (1 - t) / (1 + t)$, or

$$e = \sin \varphi; \quad e^2 = \sin \varphi^2 = \tan ^{4} \frac{1}{4} \varphi; \quad t = \tan \frac{1}{2} \varphi^2 = \sqrt{e^2}.$$

For reference, $F(e, \frac{1}{4} \pi) = (1 + e^2) F(e', \frac{1}{4} \pi), \quad 1 + e^2 = (1 + t^2) / (1 + t^2)$

$$A = (1 + t^2)^{\frac{1}{2}} [1 + \left( \frac{1}{2} \right)^{2} + \left( \frac{\varphi}{2} \right)^{2} \tan ^{4} \frac{1}{4} \varphi + \left( \frac{\varphi}{2} \right)^{3} \tan ^{12} \frac{1}{12} \varphi + \ldots].$$

When $e = .9851714 = \sin 80^\circ 07' 15'', \quad \varphi^2 = 45^\circ, \quad t = \tan \frac{221^\circ}{4}; \quad t^4 = .000 000 004. \quad \text{The first omitted term is .000 000 0318; so that the written terms would give the value of $A$ exact to ten places.}$

The common logarithms of the series of squares of binomial coefficients are, $\log \left( \frac{3}{2} \right)^{2} = 1.39794 00087; \quad \log \left( \frac{5}{2} \right)^{2} = 1.14806 25534; \quad \log \left( \frac{7}{2} \right)^{2} = 2.98970 00434; \quad \log \left( \frac{9}{2} \right)^{2} = 2.87371 61496; \quad \log \left( \frac{11}{2} \right)^{2} = 2.78220 11684; \quad \log \left( \frac{13}{2} \right)^{2} = 2.70662 40466.$

VI. Inverse Method of Integration. Referring to the close of Section II, let the longest side of the spherical triangle $c = 90^\circ$. The other two sides $a, b, \theta, \theta'$, termed complementary, are then connected by the equations,

$$F(e, \theta) + F(e, \theta') = F(e, \frac{1}{4} \pi); \quad \text{or} \quad A(a + x') = A. \frac{1}{2} \pi;$$

$$0 = \cos \theta \cos \theta' - \sin \theta \sin \theta' \sqrt{(1 - e^2)}.$$

If $A = \sqrt{(1 - e^2 \sin^2 \theta)}$, $e \sin \theta = \sqrt{(1 - e^2)}$, $e \cos \theta = \sqrt{(e^2 - 1 + e^2)}$; and
so for \( \theta', A' \). Multiply the preceding equation by \( \varepsilon' \), substituting in terms of \( A, A' \), and reducing, since \( A' = \sqrt{(1 - \varepsilon^2 \sin^2 \theta')} \)

\[
A' = \sqrt{(1 - \varepsilon^2)} = b.
\]

This constancy of the complementary product leads to the preliminary assumptions, \( A = \sqrt{b \cdot f \cos 2\theta}, A' = \sqrt{b \cdot f \cos 2\theta'} \). Here \( f \) must denote such a function of \( \cos 2\theta \) that the product \( f \cos 2\theta \cdot f \cos 2\theta' = 1 \); or one function is equal to the reciprocal of the other. And since \( 2dF = \Delta d(2x) = d(2\theta) \pm A' \), and by the Calculus, \( d(2x) \) is \(-d \cos 2x + \sqrt{1 - \cos^2 2x} \), it follows that \( f \cos 2\theta \) is \( f' \cos 2x \). Changing then to \( 2x \), since \( 2x' = \pi - 2x \), as above indicated, and using the data of \( f' \), we have then \( f' \) must denote such a function that \( f' \cos 2x \) is the reciprocal of \( f' \cos (\pi - 2x) \). This condition is fulfilled by assuming

\[
\frac{\sqrt{(1 - \varepsilon^2 \sin^2 \theta)}}{\sqrt{b}} = \frac{1+a_1 \cos 2x + a_2 \cos 4x + a_3 \cos 6x + a_4 \cos 8x + \ldots}{1-a_1 \cos 2x + a_2 \cos 4x - a_3 \cos 6x + a_4 \cos 8x + \ldots}
\]

Or adding and subtracting the identical equation \( 1 = 1 \), and dividing the one result by the other; also denoting the quotient by \( S \),

\[
S = \frac{A - \sqrt{b}}{A + \sqrt{b}} = \frac{1+a_1 \cos 2x + a_2 \cos 4x + a_3 \cos 6x + a_4 \cos 8x + \ldots}{1 + a_1 \cos 2x + a_2 \cos 4x + a_3 \cos 6x + a_4 \cos 8x + \ldots}
\]

Multiplying both sides by the right hand denominator, differentiating and dividing by \(-dx\); since \( d\theta = A \Delta dx \),

\[
\frac{A - \sqrt{b}}{A + \sqrt{b}} \left( 4a_2 \sin 4x + 8a_4 \sin 8x + \ldots \right) + \frac{2e^2 \sin \theta \cos \theta \cdot A \sqrt{b}}{(A + \sqrt{b})^2} \times
\]

\[
(1 + a_1 \cos 2x + a_2 \cos 4x + a_3 \cos 6x + \ldots) = 2a_1 \sin 2x + 6a_2 \sin 6x + \ldots
\]

In the last two equations, omitting all terms of the series after \( a_3 \), let us substitute the correlative values, firstly \( x = 0, \theta = 0, \) or \( A = 1 \). The former equation gives, since, by Section \( V \), \( \sqrt{b} = (1-t) + (1+t) \),

\[
a_1 - ta_2 + a_3 - ta_4 = t.
\]

But the derivative or latter equation vanishes. Secondly, the correlative values \( x = 45^\circ \) or \( 2x = 90^\circ, \ A = A' = \sqrt{b} \), cause the former equation to vanish, while the latter becomes, since \( e \sin \theta = \sqrt{(1 - \varepsilon^2)} \), etc.,

\[
a_1 + \frac{1}{2} A(1 - b)a_2 - 3a_3 - \frac{1}{4} A(1 - b)a_4 = + \frac{1}{2} A(1 - b).
\]

Approximately, omitting \( a_3, a_4 \), and taking \( A, b \) in terms of \( t \) from Section VI, we find from the last two equations,

\[
a_1 = t + \frac{1}{2} e^\varepsilon + \ldots, \ a_2 = \frac{1}{8} e^\varepsilon + \ldots
\]

Again differentiating, and substituting in place of \( x \) and \( \theta \) or \( A \), as before, we find a third equation; from which and the two former, omitting \( a_4 \), we find by elimination, exact to another power of \( t \),

\[
a_1 = t + \frac{1}{8} e^\varepsilon + \frac{1}{2} \varepsilon e^\varepsilon + \ldots, \ a_2 = \frac{1}{8} e^\varepsilon + \frac{1}{8} \varepsilon e^\varepsilon + \ldots, \ a_3 = \frac{1}{8} e^\varepsilon + \ldots
\]

By repeated differentiations, other equations can be found in the same way, and the approximation carried to any extent, disclosing this remark-
able law; let \( t = (1 - \sqrt{b})/(1 + \sqrt{b}) \); \( q = \frac{1}{4} t + \frac{1}{16} t^2 + \frac{1}{64} t^3 + \frac{1}{128} t^4 + \cdots \); then \( a_1 = 2q, a_2 = 2q^2, a_3 = 2q^3, a_4 = 2q^4, \ldots \). This has been fully demonstrated in a different way by its illustrious discoverer, Jacobi. He has given in the twenty-sixth volume of Crelle's Journal an extended table of the values of \( q \) for every \( 6' \) of \( \varphi \). In this, and in other ways may be demonstrated one of the most important results, the inverse relation,

\[
\frac{\sqrt{1 - e^2 \sin^2 \theta}}{\sqrt{b}} = \frac{1 + 2q \cos 2x + 2q^2 \cos 4x + 2q^3 \cos 6x + \cdots}{1 - 2q \cos 2x + 2q^2 \cos 4x - 2q^3 \cos 6x + \cdots}
\]

In the next place, for determining the integral \( x \) through \( \cos 2x \), by reversion of series, we take the second form, found before,

\[
S = \frac{d - \sqrt{b}}{d + \sqrt{b}} = \frac{2q \cos 2x + 2q^2 \cos 6x + \cdots}{1 + 2q \cos 4x + 2q^2 \cos 8x + \cdots}
\]

Multiplying both sides by the right hand denominator, and changing \( \cos 4x, \cos 6x, \text{ etc.} \), to powers of \( \cos 2x \), also omitting the sixteenth and higher powers of \( q \).

\[
S(1 - 2q^2 + 4q^4 \cos^2 2x) = (2q - 6q^3) \cos 2x(1 + 4q^4 \cos^2 2x).
\]

Dividing both members by the right hand factor, omitting \( q^{12} \),

\[
S[1 - 2q^2 + 4(q^2 - q^4) \cos^2 2x] = (2q - 6q^3) \cos 2x.
\]

This equation, being a common quadratic, is easily resolved through an auxiliary arc \( v \); the well known form of the result will be, when \( P \) and \( Q \) are functions of \( q \) only, \( \sin v = PS; \cos 2x = Q \tan \frac{v}{2} \).

To determine \( P \) and \( Q \) independently of \( q \), we first eliminate \( v \), since \( \sin v = 2 \tan \frac{v}{2} \tan \frac{v}{4} + (1 + \tan^2 \frac{v}{4}) \), and in the first and third derivatives of the result, we make \( x = 45^\circ \), and \( d = \sqrt{b} \), which gives, if \( \sin \varphi = e, \cos \varphi = b; t = (1 - \sqrt{b})/(1 + \sqrt{b}) \), the practical solution,

\[
Q = \sqrt{\left[ \frac{48}{A^2(1 + 6b + b^2)} \right] - \frac{8}{9} - \frac{16}{4} \frac{1}{9} \frac{1}{2} \cdots}
\]

\[
\sin v = \frac{48}{(1 + b)AQ} \frac{d - \sqrt{b}}{d + \sqrt{b}}; \cos 2x = Q \tan \frac{v}{2}.
\]

The resulting integral \( x \) should be exactly to \( q^{12} \), that is to ten decimal places when \( \varphi \) is less than \( 70^\circ \), or exact to seven decimals when \( \varphi \) is less than \( 83^\circ \). The auxiliary \( S \) varies from the limit \(-t\) to \(+t\); so that \( v \) may be positive, 0, or negative; and \( 2x \) greater or less than \( 90^\circ \) or \( \frac{1}{4} \pi \). When \( \varphi \) is near to \( 90^\circ \), the value of \( x \) may be found from the standard equation by the process of trial and error, to any degree of accuracy. Finally \( F = A \times x \).

(To be concluded in No. 1, Vol. V.)
EVLUTE TO CURVE OF LOGARITHMIC SINES.

BY PROF. L. G. BARBOUR, RICHMOND, KENTUCKY.

In the figure, $Or'p'n'$ and the symmetrical branch $C't'v'$ &c., represent the involute as treated in a previous article; and now $\mu\nu\tau$ and the symmetrical branch $\mu\delta\nu\tau$ represent the evolute.

An involute may be regarded as generated by a point at the extremity of a straight line every where normal to the curve; and the evolute as generated by the other extremity of the same straight line. Usually the length of this line varies according to the law that it shall be equal to the radius of curvature of the involute, though some other law might be assigned.

In this instance the radius of curvature,

$$R = \sqrt[3]{1 - x^2 + x^4} \cdot \frac{1}{1 - x^4}.$$

Maxima and Minima.

$$\frac{dR}{dx} = \frac{-2x^3 - x^5 + 10x^3 - 3x}{(1 - x^4)^{\frac{3}{2}}} = \infty \text{ or } 0.$$  

It will equal $\infty$ when $1 - x^4 = 0$. This equation has four roots, $+1, -1, +\sqrt{2} - 1, -\sqrt{2} - 1$. Using the two real roots, $x = \pm 1$, we find $R = \infty$. As the involute is always contained within the tangents $AE$ and $DF$ which are its asymptotes, the evolute has two infinite branches, and the two corresponding radii of curvature may be said to coincide at an infinite distance below the origin of coordinates.

But let the numerator, $-2x^3 - x^5 + 10x^3 - 3x$, $= 0$. One root is evidently 0. To test whether it is a maximum or minimum,

$$\frac{d^2R}{dx^2} = -14x^5 - 5x^4 + 30x^3 - 3.$$  

Substituting $x = 0$, we get $-3$; $\therefore x = 0$ gives a maximum value. As the involute is tangent to axis of abscissas at $C$, the radius of curvature is $C\mu$, the continuation of the line $BC$. Making $x = 0$ in

$$R = \sqrt[3]{1 - x^2 + x^4} \cdot \frac{1}{1 - x^4},$$

we find $C\mu = 1 - CB$. Hence the initial point of the evolute is at the end of the diameter $B\mu$, and the initial length of the radius of curvature equals the radius of the circle.
Dividing the equation \(-2x^7 - x^6 + 10x^5 - 3x - 0\) by \(x - 0\), we eliminate that root, and get after reduction, \(x^6 + \frac{1}{2}x^5 - 5x^4 + \frac{3}{2} = 0\). Let \(x^2 = v\). Then \(v^2 + \frac{1}{2}v^5 - 5v + 1.5 = 0\). By Sturm's Theorem, there are three real roots, viz., one between \(-2\) and \(-3\); one between \(+1\) and \(+2\); and one between \(0\) and \(+1\). By the original conditions of the problem, \(x\) can never be greater than \(1\). Then by Horner's method \(v \cdot x^3 = 0.316336\), which is a trifle too large, but a very close approximation. Therefore \(x = \pm 0.562437\). This is the abscissa of the point on the involute. It gives the function a minimum value. Radius of curvature \(= 0.770961\). Hence the arc \(\mu v\) or \(\mu_3\), its equal, is convex to axis of abscissas, and \(1 - 0.770961 = 0.229039\). 

To find \(vi\) and \(Ci\): \(i\) is a multiple point where the two branches intersect on the axis \(C\mu\). We must now introduce the equation of the locus of the centres of the osculating circles. Let \(a\) represent the abscissa and \(b\) the ordinate of the centre of any osculatrix referred to the same origin that we have been using, at \(C\):

\[
y - b = - \frac{dx^2 + dy^2}{dx^2 + dy^2}; \quad \therefore \quad b = y - \frac{1 - x^2 + x^4}{1 + x^2}, \quad a = \frac{2x^2 - x^4}{1 - x^4}.
\]

At the point \(i\), \(a = 0\), \(\therefore \quad 2x^2 - x^4 = 0\). Three of the roots are zero, \(\therefore \quad 2x^2 = 1, x = \pm \sqrt{\frac{1}{2}}\) and \(x^4 = \frac{1}{2}\). Remembering that \(y = \frac{1}{4}\log(1 - x^2)\), we find \(b = \frac{1}{2}\log\left(\frac{1}{2}\right) - \frac{1}{2} = Ci = 0.8465736\).

To get \(vi\), substitute for \(x^2\) its value \(\frac{1}{2}\) in the equation of the radius of curvature and we find for the point \(i\), R. C. = \(\sqrt{\frac{1}{2}} = \cos 30^\circ = 0.866025\). Subtracting from this the minimum value \(0.770961\), we get \(vi = 0.095064\).

Co-ordinates of \(v\). The abscissa of the corresponding point on the involute, as given above, is \(x = 0.562437\); \(\therefore \) by substituting, \(a = 0.0726197\); \(b = 0.785534\).

Length of arc \(\nu \tau\) or \(\lambda \pi\). In this case \(a = -1\), \(\therefore \quad \frac{2x^2 - x^4}{1 - x^4} = -1\); whence \(x = 0.886\). Radius of curvature, at \(\tau\) and \(\pi\), = 1.974699. Subtracting as before the minimum value \(0.770961\), we have \(\nu\tau = 1.203738\), \(\nu\tau - vi = 1.108674\). Also \(b = D\tau = 1.234227\).

Tangents, Normals &c. The tangent of the angle made by the tangent line to the evolute with the axis of ordinates is equal to \(\frac{da}{db} = \frac{x}{1 - x^2}\), as may be seen by differentiating, reducing and dividing. The tangent of the angle made by the tangent to the involute = \(-(1 - x^2) + x\), therefore the tangent of the angle made by the normal to the evolute = \(x + (1 - x^2)\). Hence the tangent to the evolute is normal to the involute. This is a particular case of the general principle, and tests the accuracy of our calculations.
At the point $\nu$, where the radius of curvature is a minimum, $x = 662437$. Therefore the tangent passing through this point and common to the two branches of the evolute, makes with the axis $Cy$ an angle $= 39^\circ 26' 16''$.

The subnormal of the involute $= x(dx + dy) = 1 - x^2$. For $x = 0$, sub-normal $= 1 =$ radius of curvature. For $x = 1$, sub-normal $= 0$, which shows that the normal is perpendicular to axis of ordinates at limit.

The normal $= \sqrt{(1-x^2 + x^4)}$. For $x = 0$, normal $= 1$; for $x = 1$, normal $= 1$.

*Scholium.* The discussion of these two correlated lines, the curve of logarithmic sines and its evolute, suggests an interesting conception of curvature, which may be made the basis of classification. The radius of curvature may be of constant or variable length.

(a). If the radius be constant, we have the circle.

(b). The radius may vary between finite limits, as in the ellipse, where the limits are $b^2/a$ and $a^2/b$; $a$ and $b$ denoting the semi-major and semi-minor axes.

(c). One limit may be zero, and the other a finite quantity, as in the cycloid. In this case the initial point of the evolute is on the involute. The radius of curvature equals twice the normal. Hence the limits are 0 and twice the diameter of the generating circle.

(d). One limit may be finite, and the other infinitely large, as in the spiral of Archimedes, the parabola, and the hyperbola, the logarithmic curve and the curve of logarithmic sines. The peculiarity of the last mentioned is that the radius of curvature does not increase steadily but at first diminishes until it reaches a minimum and then goes on increasing to infinity.

To these may be added the Lemniscate of Bernoulli and the Folium Cartesii.

(c). One limit may be zero, and the other $\infty$, as in the logarithmic spiral. Here again the initial point of the evolute is on the involute, just as in the cycloid, and it is noteworthy that, in both cases, the evolutes are curves precisely equal to their involutes, and differing only in position.

Another needful element of the conception is that the radius of curvature revolves through a greater or a smaller arc. In the circle the ellipse, and if we make a re-entrant curve of two cycloids, or four curves of pursuit, in their cases also, radius of curvature revolves through $360^\circ$. In the parabola, and the curve of logarithmic sines, through $180^\circ$. In the hyperbola, the arc equal $180^\circ$ less the angle which the asymptotes make with each other. In the spirals the arc is unlimited.
Quaternions.

By Christine Ladd, Union Springs, New York.

Trigonometry.—Hamilton does not give the quaternion equivalents for all the transformations of Trigonometry, and those which he does give are widely scattered through his works. I have found a résumé of the subject fruitful in suggesting new relations between the symbols of quaternions.

The following formulæ from the Elements will be made use of:—

210. XI. \[ S.q^2 = (Sq)^2 + (Vq)^2. \]

XII. \[ V.q^2 = 2Sq.Vq. \]

XV. \[ T.q^2 = (Sq)^2 - (Vq)^2. \]

XVII. \[ SU(q)^2 = 2(SUq)^2 - 1. \]

XVIII. \[ Sq'q = Sq'.Sq + S(Vq'.Vq). \]

XIX. \[ Vq'q = Vq'.Sq + Vq.Sq' + V(Vq'.Vq). \]

274. XII. \[ (TV:S)V.q = V[(Tq - Sq) - (Tq + Sq)]. \]

XII. \[ TV\sqrt{q} = \sqrt{\frac{1}{2}(Tq - Sq)}. \]

199. \[ S\sqrt{q} = \sqrt{\frac{1}{2}(Tq - Sq)}. \]

205. \[ V\Sigma = V, \Sigma^2 = \Sigma S. \]

294. II. \[ V.V\beta\alpha = \alpha S\beta\gamma - \beta S\gamma\alpha. \]

210. XXIX. \[ Tq' + Tq = T(q' + q) \text{ if } q' = xq. \]

\[ S.V\beta\alpha = \gamma \beta\alpha - \gamma S\beta\alpha + \beta S\gamma\alpha - \alpha S\gamma. \]

We have, for the trigonometrical functions,

\[ \sin \angle q = TVUq, \quad \cos \angle q = SUq, \]

\[ \sin n \angle q = TVU(q^n), \quad \cos n \angle q = SU(q^n), \]

\[ \tan \angle q = (TV:S)q. \]

Putting \( Uq \) for \( q \) in (1) and taking the tensor,

\[ SU(q^2) = (SUq)^2 + (TVUq)^2, \]

\[ \cos 2x = \cos^2x - \sin^2x; \quad \text{or from (4)} \]

\[ \cos 2x = 2\cos^2x - 1. \]

By treating (2) in a similar way we have

\[ \sin 2x = 2\sin x \cos x. \]

From (3)

\[ 1 = (SUq)^2 - (VUq)^2, \]

\[ 1 = \sin^2x + \cos^2x. \]

Introducing in (6) the condition that \( q' \) and \( q \) be coplanar, and substituting versors, we have \( VUq'q = VUq'SUq + VUq'SUq'. \)

Taking the tensor of this equation and observing that \( VUq' \parallel VUq \), we have, by (12),

\[ TVUq'q = TVUq'SUq + SUq'TVUq \]

\[ \sin(x + y) = \sin x \cos y + \cos x \sin y. \]

We have from (5), since
\[ S(Vq', Vq) = -TVq' TVq \cos \angle(Vq': Vq) = -TVq'. TVq, \]
\[ SUq'q = SUq'q SUq - TVUq' TVUq, \]
\[ \cos(x + y) = \cos x \cos y - \sin x \sin y. \]

Substituting \( q'q^{-1} \) for \( q \) in the last two equations,
\[ TVUq'q^{-1} = TVUq'q SUq - SUq'q TVUq, \]
\[ SUq'q^{-1} = SUq'q SUq + TVUq' TVUq, \]
which gives the values for the sine and cosine of the difference of two angles.

Adding (22) and (24), \( TVUq'q + TVUq'q^{-1} = 2SUq'q TVUq'q S \).

Putting \( q'q^{-1} = r, q'q = r', q' = \sqrt{r'r}, q = \sqrt{rr'} \) and we have
\[ TVU_r' + TVU_r = 2SU_r'(r'r^{-1}) TVU_r'(r'r), \]
\[ \sin x + \sin y = 2\sin \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y). \]

Similarly,
\[ TVU_r' - TVU_r = 2SU_r'(r'r) TVU_r'(r'r^{-1}), \]
\[ SU_r' + SU_r = 2SU_r'(r'r) SU_r'(r'r), \]
\[ SU_r' - SU_r = -2TVU_r'(r'r) TVU_r'(r'r^{-1}). \]

From (8) \[ 2(TVU_r'/q)^2 = TUq - SUq, \]
\[ 2\sin^2 \frac{1}{2}x = 1 - \cos x. \]

From (9) \[ 2(SU_r'/q)^2 = TUq + SUq, \]
\[ 2\cos^2 \frac{1}{2}x = 1 + \cos x. \]

\[ (TV:S)q^2 = \frac{28qTV_q}{(S_q)^2 + (V_q)^2} = \frac{2TV_q}{S_q} (S_q)^2 - (T_q)^2 = \frac{2(TV:S)q}{1 - [(TV:S)q]^2} \]
\[ \tan 2x = \frac{2 \tan x}{1 - \tan^2 x} \]

\[ (TV:S)q'q = \frac{(TV:S)q + (TV:S)q'}{1 - (TV:S)q(TV:S)q'}, \]
\[ (TV:S)q'q^{-1} = \frac{(TV:S)q - (TV:S)q'}{1 + (TV:S)q(TV:S)q'}, \]
\[ \tan (x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}. \]

\[ (TV:S)q' + (TV:S)q = \frac{TVUq'_q}{SUq'_q SUq}, \]
\[ (TV:S)q' - (TV:S)q = \frac{TVUq'_q}{SUq'_q SUq}, \]
\[ \tan x \pm y = \frac{\sin (x \pm y)}{\cos x \cos y}. \]

\[ (TV:S)v'(r') = \frac{TVU_r' + TVU_r}{SU_r' + SU_r}, \]
\[ (TV:S)v'(r'^{-1}) = \frac{TVU_r' - TVU_r}{SU_r' + SU_r}, \]
\[ \tan \frac{1}{2}(x \pm y) = \frac{\sin (x \pm y)}{\cos x \cos y}. \]

By (6) we have
\[ V(q''q'q) = V(q''q'q'' + q'q'q Vq'') + V(Vq''q'q) \]
\[ = Vq''q'q'' + Vq''q'q'q'' + q''q Vq'q + Vq''q'q'q + Vq''q'q'q'' + Vq''q'q'q'' + Vq''q Vq', \]

\[ + Vq''q'q'' + V(Vq''q'q'q'' + Vq''q'q'q'' + Vq''q'q'q'q'' + Vq''q'q'q'q'q''). \]
which becomes, by (10) and (11) and by arranging symmetrically,

\[ Vq^i q^i q =: Vq S_q S_q^i + Vq S_q S_q + Vq^i S_q S_q + S_q Vq Vq^i Vq \]

\[ - S_q Vq Vq Vq^i + S_q^i Vq Vq^i + Vq S_q Vq Vq^i - Vq S_q Vq Vq^i + Vq S_q^i Vq \]

(39)

By (5) \( S(q q q) = S(q S_q q + S Vq Vq) = S(q S_q S_q + S_q S_q Vq Vq) \]

\[ + S[Vq Vq S_q Vq + Vq Vq S_q Vq + Vq Vq Vq Vq], \]

or by (10) and (13), \( S(q q q) = S(q S_q S_q + Vq Vq Vq + S_q S_q Vq Vq - Vq^i S_q S_q Vq Vq \]

\[ + S_q S_q Vq Vq + Vq S_q Vq Vq + S_q S_q Vq Vq - Vq^i S_q S_q Vq Vq \]

(40)

Making the quaternions complanar and their tensors equal, and then taking the tensors of the two equations, we have for the sine and cosine of the sum of three angles,

\[ T V Y q^i q^i q = T V Y q^i S Y q S Y q + S Y q^i T V Y q S Y q + S Y q^i S Y q^i T V Y q \]

\[ - T V Y q^i T V Y q T V Y q. \]  

(41)

\[ S Y q^i q^i q = S Y q^i S Y q S Y q - T V Y q^i T V Y q S Y q \]

\[ - T V Y q^i S Y q T V Y q - S Y q^i T V Y q T V Y q. \]  

(42)

These equations might have been deduced directly from (22) and (23), but the formulae for non-complanar quaternions, (39) and (40), are not without value.

From (22), (23), (20), (2), (41) and (42), we obtain

\[ T V Y q^i q^i + T V Y q^i q^i - T V Y q^i q^i - T V Y q^i q^i q^i = 4 T V Y q^i q^i T V Y q^i q^i q^i; \]  

(43)

\[ S Y q^i q^i + S Y q^i q^i + S Y q^i q^i q^i + S Y q^i q^i q^i = 4 S Y q^i q^i S Y q^i q^i q^i; \]  

(44)

\[ - T V Y q^i q^i + T V Y q^i q^i + T V Y q^i q^i + T V Y q^i q^i q^i = 4 S Y q^i q^i T V Y q^i q^i q^i; \]  

(45)

\[ - S Y q^i q^i + S Y q^i q^i - S Y q^i q^i q^i = 4 T V Y q^i q^i S Y q^i q^i T V Y q^i q^i q^i. \]  

(46)

When \( \angle q^i q^i q = 2n \frac{1}{2} \pi \), \( T V Y q^i q^i q^i q^i = 0 \) and \( S Y q^i q^i q^i q^i = -1 \).

The above equations then become

\[ \pm 4 T V Y q^i q^i T V Y q^i T V Y q^i = T V Y q^i + T V Y q^i q^i + T V Y q^i q^i, \]

\[ \pm 4 S Y q^i S Y q^i S Y q^i \quad S Y q^i q^i = S Y q^i + S Y q^i q^i + S Y q^i q^i q^i + 1, \]

\[ \pm 4 S Y q^i S Y q^i T V Y q^i T V Y q^i = T V Y q^i + T V Y q^i q^i - T V Y q^i q^i, \]

\[ \pm 4 T V Y q^i T V Y q^i S Y q^i T V Y q^i = S Y q^i q^i = S Y q^i + S Y q^i q^i - S Y q^i q^i q^i = 1, \]

the upper sign being taken when \( n \) is even, the lower when \( n \) is odd.

When \( \angle q^i q^i q = (2n + 1) \frac{1}{2} \pi, \) \( T V Y q^i q^i q^i q^i = 0 \) and \( S Y q^i q^i q^i q^i = 1 \), and the equations become

\[ \pm 4 S Y q^i S Y q^i S Y q^i q^i = T V Y q^i + T V Y q^i q^i + T V Y q^i q^i, \]

\[ \pm 4 T V Y q^i T V Y q^i q^i q^i = S Y q^i q^i + S Y q^i q^i + S Y q^i q^i q^i + 1, \]

\[ \pm 4 T V Y q^i T V Y q^i S Y q^i q^i = S Y q^i q^i = S Y q^i + S Y q^i q^i - S Y q^i q^i q^i = 1, \]

according as \( n \) is even or odd.

---

*For want of sorts the Greek Upsilon is here used instead of U.—Compositor.
INTEGRATION OF FIVE DIFFERENTIAL FORMS.

BY ARTEMAS MARTIN, M. A., ERIE, PA.

The following forms, which sometimes occur in the solution of problems, are not integrated in any of the text-books on the Calculus to which I have access.

1. To integrate

\[
dI_1 := \frac{xdx}{(m^2 - x^2)(n^2 - x^2)^{\frac{3}{2}}}
\]

Let \( m^2 - x^2 = y^2 \), then \( xdx = -ydy \), \( n^2 - x^2 = y^2 - m^2 + n^2 \) and

\[
I_1 = \int \frac{-dy}{\sqrt{(y^2 - m^2 + n^2)}} = \frac{1}{\sqrt{(m^2 - n^2)}} \sin^{-1} \left( \frac{\sqrt{(m^2 - n^2)}y}{y} \right),
\]

\[
= \frac{1}{\sqrt{(m^2 - n^2)}} \sin^{-1} \left( \frac{\sqrt{(m^2 - n^2)} \frac{1}{2}}{\sqrt{(m^2 - x^2)}} \right), \text{ when } m > n ; \text{ and }
\]

\[
= \frac{1}{\sqrt{(n^2 - m^2)}} \log \left( \frac{\sqrt{(n^2 - m^2)} + \sqrt{(y^2 - m^2 + n^2)}}{y} \right)
\]

\[
= \frac{1}{\sqrt{(n^2 - m^2)}} \log \left( \frac{\sqrt{(n^2 - m^2)} + \sqrt{(n^2 - x^2)}}{\sqrt{(m^2 - x^2)}} \right), \text{ when } m < n.
\]

2. To integrate

\[
dI_2 := \frac{dx}{(m^2 - x^2)(n^2 - x^2)^{\frac{3}{2}}}
\]

Let \( n^2 - x^2 = x^2 y^2 \), then \( x = \frac{n}{\sqrt{1 + y^2}} \), \( y = \frac{\sqrt{(n^2 - x^2)}}{x} \), \( dx = \frac{-nydy}{(1+y^2)^\frac{3}{2}} \).

\[
\sqrt{(n^2 - x^2)} = xy = \frac{ny}{\sqrt{1+y^2}}, \quad m^2 - x^2 = m^2 - n^2 - y^2 = \frac{m^2 - n^2 + m^2 y^2}{1+y^2};
\]

\[
\therefore I_2 = \int \frac{-nydy}{(1+y^2)^\frac{3}{2}} \times \frac{1 + y^2}{m^2 - n^2 + m^2 y^2} \times \frac{\sqrt{(1+y^2)}}{ny},
\]

\[
= \int \frac{-dy}{m^2 y^2 + m^2 - n^2} = \frac{1}{m \sqrt{(m^2 - n^2)}} \tan^{-1} \left( \frac{\sqrt{(m^2 - n^2)}}{ny} \right),
\]

\[
= \frac{1}{m \sqrt{(m^2 - n^2)}} \tan^{-1} \left( \frac{x \sqrt{(m^2 - n^2)}}{\sqrt{(m^2 - x^2)}} \right),
\]

\[
= \frac{1}{m \sqrt{(m^2 - n^2)}} \sin^{-1} \left( \frac{x \sqrt{(m^2 - n^2)}}{n \sqrt{(m^2 - x^2)}} \right), \text{ when } m > n;
\]

and

\[
= \frac{1}{2m \sqrt{(m^2 - n^2)}} \log \left( \frac{my + \sqrt{(n^2 - m^2)}}{\sqrt{(n^2 - m^2)}} \right)
\]

\[
= \frac{1}{m \sqrt{(m^2 - n^2)}} \log \left( \frac{m \sqrt{(n^2 - x^2)} + nx \sqrt{(n^2 - m^2)}}{n \sqrt{(m^2 - x^2)}} \right), \text{ when } m < n.
\]
3. To integrate \( dI_3 = \sin^{-1} \left( \frac{b^2}{a^2 - x^2} \right)^{\frac{1}{2}} dx \).

By the method of parts,

\[
I_3 = x \sin^{-1} \left( \frac{b^2}{a^2 - x^2} \right)^{\frac{1}{2}} - \int \frac{bx^2 dx}{(a^2 - x^2)(a^2 - b^2 - x^2)^{\frac{1}{2}}}.
\]

But \(- \int \frac{bx^2 dx}{(a^2 - x^2)(a^2 - b^2 - x^2)^{\frac{1}{2}}} = \int \frac{b dx}{(a^2 - b^2 - x^2)^{\frac{1}{2}}} + \int \frac{a^2 b dx}{(a^2 - x^2)(a^2 - b^2 - x^2)^{\frac{1}{2}}}\),

\[
\therefore I_3 = x \sin^{-1} \left( \frac{b^2}{a^2 - x^2} \right)^{\frac{1}{2}} + b \sin^{-1} \left( \frac{x^2}{a^2 - b^2} \right)^{\frac{1}{2}} - a \sin^{-1} \left( \frac{b^2 x^2}{(a^2 - b^2)(a^2 - x^2)^{\frac{1}{2}}} \right)^{\frac{1}{2}}.
\]

4. To integrate \( dI_4 = \sin^{-1} \left( \frac{x^2}{a^2 - x^2} \right) dx \).

\[
I_4 = x \sin^{-1} \left( \frac{x^2}{a^2 - x^2} \right) - \int \frac{2ax^2 dx}{(a^2 - x^2) (a^2 - 2x^2)^{\frac{1}{2}}}.
\]

\[
= x \sin^{-1} \left( \frac{x^2}{a^2 - x^2} \right) + ax - \int \frac{dx}{(a^2 - x^2)^{\frac{1}{2}}} - \frac{a^2}{2} \int \frac{dx}{(a^2 - x^2)^{\frac{1}{2}}}.
\]

\[
= x \sin^{-1} \left( \frac{x^2}{a^2 - x^2} \right) + ax - 2a \sin^{-1} \left( \frac{x}{a} \right).
\]

5. To integrate \( dI_5 = \sin^{-1} \left( \frac{x^2}{a^2 - x^2} \right)^{\frac{1}{2}} dx \).

\[
I_5 = x \sin^{-1} \left( \frac{x^2}{a^2 - x^2} \right)^{\frac{1}{2}} - \frac{x dx}{(a^2 - x^2)(a^2 - 2x^2)^{\frac{1}{2}}}.
\]

\[
= x \sin^{-1} \left( \frac{x^2}{a^2 - x^2} \right)^{\frac{1}{2}} - a \sqrt{2} \sin^{-1} \left( \frac{a}{2(a^2 - x^2)} \right)^{\frac{1}{2}}.
\]

**Note.**—In reference to the paragraph at the bottom of page 141 of the *Analyst* for September, Prof. Abbe writes that when, in February, 1872, he wrote the article referred to, he desired only to bring to notice Dr. Adrain’s early and original investigations and therefore selected for re-publication that one of the two proofs which its author seems to consider most satisfactory. The omission was not accidental—nor would it have been made had he duly considered the historical points that have been subsequently developed by Glaisher and Merriman.

**Query.** (By request.)—In Salmon’s Higher Plane Curves, 2nd ed. p. 180, is found the following proposition:—“A right angle has the side GF of fixed length, the point F moves along the fixed line CI, while the side GH passes through the fixed point E, a pencil at the middle point of GF will describe the Cissoid.” How is this demonstrated?
NOTE ON EVALUATION OF INDETERMINATE FORMS.

BY PROF. WM. WOOLSEY JOHNSON, ANNAPOLIS, MD.

The following considerations are sometimes useful in facilitating the evaluation of functions which take an indeterminate form when the independent variable vanishes. This virtually includes all indeterminate forms; since if \( f(x) \) takes an indeterminate form when \( x = a \), we may put \( x = a = x \), and if it takes an indeterminate form when \( x = \infty \), we may put \( 1 + x = x \), in either case we have a function of \( x \) which takes a corresponding form when \( x = 0 \).

In the first place it is evident that we may decompose the function into parts and evaluate them separately, observing only that, if more than one of these parts turn out to be infinite in value, they must be recombed, since we are liable to fall upon the indeterminate form \( \infty - \infty \).

Again, we can decompose the function into factors, and evaluate them separately, observing only that, if any factor turns out to be either zero or infinite, it must be treated in connection with the other zero and infinite factors. Now transferring the infinite factors to the denominator we have to deal with a fractional expression composed of zero factors in both terms.

Again, we may substitute for any zero factor any other zero factor which bears to it the ratio of equality when \( x = 0 \); for in so doing we merely multiply the function by a factor which takes the form \( \frac{0}{0} \) but is known to have the value unity when \( x = 0 \).

We may call these zero factors equivalents: thus by reason of the familiar results
\[
\frac{\sin x}{x} \bigg|_0 = \frac{\tan x}{x} \bigg|_0 = \frac{\log(1+x)}{x} \bigg|_0 = \frac{x^2-1}{x} \bigg|_0 = 1,
\]
we may say that each of the above zero factors is equivalent to \( x \). From these we may of course infer the equivalents of such factors as \( \sin mx \bigg|_0 \), \( \tan x^2 \bigg|_0 \), \( \log(1-x) \bigg|_0 \) \&c; again, since \( (1+\cos x)(1-\cos x) \bigg|_0 = 1-\cos^2 x \bigg|_0 = \sin^2 x \bigg|_0 = x^2 \bigg|_0 \) we have \( 1-\cos x \bigg|_0 = \frac{1}{2}x^2 \bigg|_0 \).

Now if we substitute for each zero factor its equivalent in the form of a power of \( x \), the value of the function is at once apparent.

The equivalents of the elementary forms will readily occur to the memory, and moreover, when the developments of the constituents of a zero factor are known, the equivalent which is simply the lowest term in the resulting development is readily inferred, thus the equivalent of the factor \( x - \sin x \bigg|_0 = \frac{1}{2}x^2 \bigg|_0 \).
To find the equivalent of a zero factor \( f(x) \) we may employ the usual method, thus, if \( f^n(x) \) is the first of the series of derivatives which does not vanish when \( x = 0 \), then
\[
\frac{f(x)}{x^n} \bigg|_0 = \frac{f^n(0)}{n!},
\]
hence the equivalent of \( f(x) \) is \( f^n(0) \frac{x^n}{n!} \), which is the first term of the development of \( f(x) \) by Maclaurin's Theorem.

In finding the equivalent of a polynomial factor we may avail ourselves of the equivalents of its elements, the equivalent of the whole being the lowest term resulting from the substitution; remembering however that, if the terms containing the lowest power of \( x \) have a vanishing coefficient these terms must still be considered in determining the equivalent, which is thus seen to be of a higher order. Thus if the factor is \( \log (1+x) - x + x^2 \) since the equivalent of \( \log (1+x) \) is \( x \) the terms of the first degree vanish; it must now be assumed that the equivalent is \( x^3 \), for it will be found that the equivalent of \( \log (1+x) - x \) is \( -\frac{1}{3}x^3 \), hence that of the whole expression is \( \frac{4}{3}x^3 \).

As an example, I apply the foregoing principles to the following which occurs in Todhunter's Diff. Calculus:

\[
\frac{(e^x - e^{-x})^2 - 2x^2(e^x + e^{-x})}{x^4} \bigg|_0.
\]

The equivalent of \( e^x - e^{-x} \) is \( 2x \), hence neither of the terms of this expression is finite, but after taking the derivative of each term we have
\[
\frac{e^{2x} - e^{-2x} - 2x^2(e^x - e^{-x})}{2x^3} - 2x(e^x + e^{-x}) \bigg|_0,
\]
of which the middle term has a finite value. Evaluating this, and repeating the derivative process for the rest, we have
\[
\frac{e^{2x} + e^{-2x} - (e^x + e^{-x}) - 2x(e^x - e^{-x})}{3x^2} \bigg|_0 = -1,
\]
which again contains a finite portion, hence proceeding as before we have
\[
\frac{2(e^{2x} - e^{-2x}) - (e^x - e^{-x})}{6x} \bigg|_0 = \frac{2}{3} - 1
\]
or
\[
\frac{8x - 2x}{6x} \bigg|_0 = \frac{2}{3} - 1 = \frac{2}{3}.
\]
ANSWER TO QUERY. (SEE PAGE 128.)

BY PROF. D. J. MC ADAM, WASHINGTON, PENNSYLVANIA.

As preliminary to an answer to Mr. Baker’s Query I submit for publication the following problem and its solution by Prof. G. B. Vose, not knowing whether it has ever been published or not.

Problem. — It is required to determine the relation which exists between the fifteen angles which six spheres make with each other.

Solution.—Let \( x_1, y_1, z_1, r_1 \) be the coordinates of the center and the radius of the first sphere, with similar expressions for the remaining spheres.

Let \((1, 2)\) be the cosine of the angle of the radii of the spheres 1 and 2, drawn to a common point. The square of the distance between the centers of 1 and 2 may be expressed in two different ways; viz.,

\[
    r_1^2 + r_2^2 - 2r_1r_2(1, 2) \text{ and } (x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2.
\]

Equating these expressions and transposing, we have

\[
    (x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2 - r_1^2 - r_2^2 = -2r_1r_2(1, 2), \tag{1}
\]

Similarly

\[
    (x_1-x_3)^2 + (y_1-y_3)^2 + (z_1-z_3)^2 - r_1^2 - r_3^2 = -2r_1r_3(1, 3), \tag{2}
\]

\[
    (x_1-x_4)^2 + \&c. = -2r_1r_4(1, 4), \tag{3}
\]

\[
    (x_1-x_5)^2 + \&c. = -2r_1r_5(1, 5), \tag{4}
\]

\[
    (x_1-x_6)^2 + \&c. = -2r_1r_6(1, 6), \tag{5}
\]

\[
    (x_2-x_3)^2 + \&c. = -2r_2r_3(2, 3), \tag{6}
\]

\[
    (x_2-x_4)^2 + \&c. = -2r_2r_4(2, 4), \tag{7}
\]

\[
    (x_2-x_5)^2 + \&c. = -2r_2r_5(2, 5), \tag{8}
\]

\[
    (x_2-x_6)^2 + \&c. = -2r_2r_6(2, 6), \tag{9}
\]

\[
    (x_3-x_4)^2 + \&c. = -2r_3r_4(3, 4), \tag{10}
\]

\[
    (x_3-x_5)^2 + \&c. = -2r_3r_5(3, 5), \tag{11}
\]

\[
    (x_3-x_6)^2 + \&c. = -2r_3r_6(3, 6), \tag{12}
\]

\[
    (x_4-x_5)^2 + \&c. = -2r_4r_5(4, 5), \tag{13}
\]

\[
    (x_4-x_6)^2 + \&c. = -2r_4r_6(4, 6), \tag{14}
\]

\[
    (x_5-x_6)^2 + \&c. = -2r_5r_6(5, 6). \tag{15}
\]

We must now eliminate the 24 unknowns \( x_1, y_1, z_1, r_1, x_2, \&c. \), and obtain a relation between \((1, 2), (1, 3), (2, 4) \&c. \). This seems at first sight to be a very tedious operation, but the theory of Determinants furnishes an easy method. The Determinant,

\[
    \begin{vmatrix}
        0 & 1 & x_1^2 + y_1^2 + z_1^2 - r_1^2 & x_1 & y_1 & z_1 \\
        0 & 1 & x_2^2 + y_2^2 + z_2^2 - r_2^2 & x_2 & y_2 & z_2 \\
        0 & 1 & x_3^2 + y_3^2 + z_3^2 - r_3^2 & x_3 & y_3 & z_3 \\
        0 & 1 & x_4^2 + y_4^2 + z_4^2 - r_4^2 & x_4 & y_4 & z_4 \\
    \end{vmatrix}
\]
\(-180-\)

| 0  1  \(x_5^2 + y_5^2 + z_5^2 - r_5^2\)  \(x_5\)  \(y_5\)  \(z_5\)  \\
| 0  1  \(x_6^2 + y_6^2 + z_6^2 - r_6^2\)  \(x_6\)  \(y_6\)  \(z_6\) |

is evidently zero because the elements of the first column vanish.

For the same reason the determinant

\[
\begin{vmatrix}
0  & x_1^2 + y_1^2 + z_1^2 - r_1^2 & 1 & -2x_1 & -2y_1 & -2z_1 \\
0  & x_2^2 + y_2^2 + z_2^2 - r_2^2 & 1 & -2x_2 & -2y_2 & -2z_2 \\
0  & x_3^2 + y_3^2 + z_3^2 - r_3^2 & 1 & -2x_3 & -2y_3 & -2z_3 \\
0  & x_4^2 + y_4^2 + z_4^2 - r_4^2 & 1 & -2x_4 & -2y_4 & -2z_4 \\
0  & x_5^2 + y_5^2 + z_5^2 - r_5^2 & 1 & -2x_5 & -2y_5 & -2z_5 \\
0  & x_6^2 + y_6^2 + z_6^2 - r_6^2 & 1 & -2x_6 & -2y_6 & -2z_6
\end{vmatrix}
\]

vanishes.

The product of these two Determinants is therefore zero. Hence

\[
0 = \begin{vmatrix}
-2r_1r_1 & -2r_2r_1(2, 1) & -2r_3r_1(3, 1) & -2r_4r_1(4, 1) & -2r_5r_1(5, 1) & -2r_6r_1(6, 1) \\
-2r_1r_2(1, 2) & -2r_2r_2 & -2r_3r_2(3, 2) & -2r_4r_2(4, 2) & -2r_5r_2(5, 2) & -2r_6r_2(6, 2) \\
-2r_1r_3(1, 3) & -2r_2r_3(2, 3) & -2r_3r_3 & -2r_4r_3(4, 3) & -2r_5r_3(5, 3) & -2r_6r_3(6, 3) \\
-2r_1r_4(1, 4) & -2r_2r_4(2, 4) & -2r_3r_4(3, 4) & -2r_4r_4 & -2r_5r_4(5, 4) & -2r_6r_4(6, 4) \\
-2r_1r_5(1, 5) & -2r_2r_5(2, 5) & -2r_3r_5(3, 5) & -2r_4r_5(4, 5) & -2r_5r_5 & -2r_6r_5(6, 5) \\
-2r_1r_6(1, 6) & -2r_2r_6(2, 6) & -2r_3r_6(3, 6) & -2r_4r_6(4, 6) & -2r_5r_6(5, 6) & -2r_6r_6
\end{vmatrix}
\]

Now divide each element of the first column by \(-2r_1\), each element of the second column by \(-2r_2\), &c. Also divide each element of the first rank by \(r_1\), each element of the second rank by \(r_2\), &c., and the Determinant becomes,

\[
\begin{vmatrix}
1 & (2, 1) & (3, 1) & (4, 1) & (5, 1) & (6, 1) \\
(1, 2) & 1 & (3, 2) & (4, 2) & (5, 2) & (6, 2) \\
(1, 3) & (2, 3) & 1 & (4, 3) & (5, 3) & (6, 3) \\
(1, 4) & (2, 4) & (3, 4) & 1 & (5, 4) & (6, 4) \\
(1, 5) & (2, 5) & (3, 5) & (4, 5) & 1 & (6, 5) \\
(1, 6) & (2, 6) & (3, 6) & (4, 6) & (5, 6) & 1
\end{vmatrix} = 0.
\]

This is the solution required. The cosines \((2, 1)\) and \((1, 2)\) are identical but written differently for the sake of symmetry.

To apply this general formula to the question under consideration; since the five spheres are given in position, the angles whose cosines are \((1, 2)\) &c., are all known except the five \((1, 6), (2, 6), (3, 6), (4, 6)\) and \((5, 6)\), and the identical angles \((6, 1)\), &c. Since these are to be equal, calling \((1, 6)\) &c. \(x\) and the known cosines, \(a, b, \&c\), our Determinant becomes,

\[
\begin{vmatrix}
1 & a & b & c & d & x \\
a & 1 & e & g & l & x \\
b & e & 1 & h & k & x \\
c & g & h & 1 & n & x \\
d & l & k & n & 1 & x \\
x & x & x & x & x & 1
\end{vmatrix} = 0.
\]
Dividing the determinant by $x^2$ by dividing the sixth column and sixth rank by $x$; also taking the lower rank to the top by five transpositions and five changes of sign; also the sixth column to the first by five additions, transpositions and changes of sign, and we get

\[
\begin{vmatrix}
  1 & 1 & 1 & 1 & 1 & 1 \\
  1 & a & b & c & d & \frac{1}{x^2} \\
  1 & a & 1 & e & g & l \\
  1 & b & e & 1 & h & k \\
  1 & c & g & h & 1 & n \\
  1 & d & l & k & n & 1
\end{vmatrix} = 0. \quad \text{Hence} \quad \begin{vmatrix}
  1 & a & b & c & d \\
  a & 1 & e & g & l \\
  b & e & 1 & h & k \\
  c & g & h & 1 & n \\
  d & l & k & n & 1
\end{vmatrix} = \begin{vmatrix}
  a & b & c & d \\
  1 & e & g & l \\
  h & k & 1 & b \\
  g & h & 1 & n \\
  k & n & 1 & 1
\end{vmatrix}
\]

From this equation $\frac{1}{x}$ is at once found, and hence $x$ and $\cos^{-1}x$.

Having the $\cos^{-1}x$, to construct the sixth sphere. With the centers of the five spheres as vertices and indefinite lines making $\cos^{-1}x$ with the radii of the respective spheres drawn toward the common origin which we have been using, describe circular cones, which will intersect in a common point which is the center of the sixth sphere.

All spheres having this point as center will cut the five spheres at equal angles.

**Note on Attraction, by R. J. Adcock, Monmouth, Ill.**—If every particle of matter attracts from all directions with an equal constant force, then the attraction between masses or molecules must vary directly as their sum and inversely as the square of their distance. That no other law is possible follows from the following considerations:

If every particle attracts with the same constant force, then, that the attraction is as the sum of the masses follows from the axiom that the whole is equal to all its parts. And if the attraction of each particle is a constant force exerted in all directions, then, obviously, because the areas over which the force is distributed at different distances vary as the squares of the distances, the energy exerted upon a point, or upon a particle of matter at any distance, is inversely as the square of the distance.

Hence, from the known laws of attraction, we have this ultimate proposition:— Assuming the ultimate particles of matter to be infinitely small, every particle attracts, or draws, from all directions, with an equal and constant force without regard to the distance of its point of application.
SOLUTION OF PROB. 174 BY CHAS. H. KUMMELL. (CONTINUED FROM P. 157.)

But \( \Gamma(1_{\frac{1}{2}} n) \) may now be likewise determined. We have by (4), p. 120,
\[
\Gamma\left(\frac{1}{2}\right) = \frac{(2\pi)^{1/2}}{I^{1/2}} = 2^{1/2} \pi^{1/2} \left\{ \sum_{n=0}^{\infty} \frac{1}{n!} \right\} \Gamma\left(\frac{1}{2}\right)^{1/2}.
\]
(11)

By theorem (2) we have
\[
\Gamma\left(\frac{1}{3}\right) = \frac{\pi}{\sin \frac{1}{3} \pi \Gamma\left(\frac{1}{3}\right)},
\]
(12)
\[
\Gamma\left(\frac{2}{3}\right) = \frac{\pi}{\sin \frac{2}{3} \pi \Gamma\left(\frac{2}{3}\right)} = \frac{2^{3/2} \pi}{2^{3/2} \pi} = \Gamma\left(\frac{2}{3}\right),
\]
(13)
\[
\Gamma\left(\frac{4}{3}\right) = \frac{\pi}{\sin \frac{4}{3} \pi \Gamma\left(\frac{4}{3}\right)} = \frac{\pi}{\sin \frac{4}{3} \pi \Gamma\left(\frac{4}{3}\right)}.
\]
(14)

\( \Gamma\left(\frac{1}{2}\right) \) being given by (11) only \( \Gamma\left(\frac{1}{2}\right) \) and \( \Gamma\left(\frac{1}{2}\right) \) must yet be determined. Placing in (6) \( n = \frac{1}{2} \) and \( r = 2 \) we have
\[
\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right) = (2\pi)^{1/2} \Gamma\left(\frac{1}{2}\right),
\]
and dividing by (7)
\[
\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right) = \frac{2^{1/2} \pi^{1/2}}{2^{1/2} \pi^{1/2}} \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)
\]
\[
= \frac{2^{3/2} \pi^{1/2}}{2^{3/2} \pi^{1/2}} \cos \frac{1}{3} \pi \Gamma\left(\frac{1}{2}\right),
\]
by (6).
(15)

Also placing in (5) \( n = \frac{1}{2} \) and \( r = 3 \), then
\[
\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{3}{2}\right) = 2 \pi \Gamma\left(\frac{1}{2}\right),
\]
and dividing by (16)
\[
\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{3}{2}\right) = \frac{2^{3/2} \pi^{1/2}}{2^{3/2} \pi^{1/2}} \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right).
\]
(16)

Multiplying (16) by (15) and extracting the square root we obtain
\[
\Gamma\left(\frac{1}{2}\right) = 2^{3/2} \pi^{1/2} \frac{1}{2} \left(\cos \frac{1}{3} \pi \right) \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right).
\]
(17)

Also dividing (16) by (17) we obtain
\[
\Gamma\left(\frac{2}{3}\right) = 3^{3/2} \pi^{1/2} \left(\cos \frac{1}{3} \pi \right)^{1/2} \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right).
\]
(18)

This together with (1), (3), (4), (6), (10), (11), (12), (13), (14), (17) determines \( \Gamma(\frac{1}{2} n) \).

The general problem to determine \( \Gamma(n) \) may now be solved with that degree of accuracy usually required. It is known now at intervals of \( \frac{1}{2} \pi \) and if we can determine it between \( r \) and \( r + \frac{1}{2} \) at smaller intervals we know \( \Gamma(n) \) in its whole extent.

We have then by Lagrange's interpolation formula
\[
\Gamma(r+n) = 12n \cdot \frac{12n-1}{1} \cdot \frac{12n-2}{2} \cdots \frac{12n-6}{6} \left[ \frac{1}{12n} \Gamma(r) - \frac{6}{12n-1} \Gamma(r+\frac{1}{2}) \right.
\]
\[
+ \frac{15}{12n-6} \Gamma(r+\frac{1}{2}) + \frac{20}{12n-3} \Gamma(r+\frac{1}{2}) + \frac{15}{12n-4} \Gamma(r+\frac{1}{2}) - \frac{1}{12n-5} \Gamma(r+\frac{1}{2}) - \frac{1}{12n-6} \Gamma(r+\frac{1}{2}) \right].
\]
(19)
Since $\Gamma(n)$ has its minimum value between $n = 1$ and $n = 2$ it is advantageous to place $r = 1$, hence

$$
\Gamma(1+n) = 12n \cdot \frac{12n-1}{1} \cdot \frac{12n-2}{2} \cdot \cdots \cdot \frac{12n-6}{6} \left[ \frac{1}{12n-1} \frac{6}{12n-2} \Gamma(\frac{1}{2}) + \frac{15}{12n-2} \frac{20}{12n-3} \Gamma(\frac{2}{3}) + \frac{15}{12n-4} \frac{6}{12n-5} \Gamma(\frac{1}{2}) + \frac{1}{12n-6} \Gamma(\frac{3}{2}) \right].
$$

(20)

$n$ being some fraction $< \frac{1}{2}$.

The following table of the gamma function was computed by the above convenient formula:

<table>
<thead>
<tr>
<th>$n$</th>
<th>log $\Gamma(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.000</td>
<td>0.0000000</td>
</tr>
<tr>
<td>1.083</td>
<td>-9.9814951</td>
</tr>
<tr>
<td>1.166</td>
<td>9.9674166</td>
</tr>
<tr>
<td>1.250</td>
<td>9.9573211</td>
</tr>
<tr>
<td>1.333</td>
<td>9.9508415</td>
</tr>
<tr>
<td>1.416</td>
<td>9.9476700</td>
</tr>
<tr>
<td>1.500</td>
<td>9.9475450</td>
</tr>
</tbody>
</table>

and these values being used in (20) for values of $n < \frac{1}{2}$ and $> 0$ will give results correct to sixth differences.

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**NOTE ON THE METHOD OF LEAST SQUARES.**

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BY R. J. ADCOCK, MONMOUTH, ILL.

When a greater number of points are given or observed than are sufficient to determine any point, line or surface, that point, line or surface which makes the sum of the squares of the errors of situation a minimum, has the greatest probability, and is therefore the one determined by these points.

(1). Let the coordinates, $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots, (x_n, y_n, z_n)$ of $n$ points, be given or measured, and represent by $d_1, d_2, \ldots, d_n$, the distances respectively from the $n$ points to any point $(a, \beta, \gamma)$.

If $m$ = the number of points on a unit of surface, then the probability that a point, taken at random on any of the surfaces of the spheres whose centers are $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots, (x_n, y_n, z_n)$, and radii respectively $d_1, d_2, \ldots, d_n$, shall be at the point $(a, \beta, \gamma)$, is $1/\pi m(d_1^2 + d_2^2 + \ldots + d_n^2)$, which probability is greatest when $d_1^2 + d_2^2 + \ldots + d_n^2 = a$ minimum.
(2). Let the coordinates \((x_1, y_1, z_1), (x_2, y_2, z_2), \ldots, (x_n, y_n, z_n)\), of \(n\) points be given or measured, and let \(\delta_1, \delta_2, \ldots, \delta_n\) be the normals respectively from the \(n\) points to any line or surface. Then the probability that a point, taken at random on any of the surfaces of the spheres whose centers are \((x_1, y_1, z_1), (x_2, y_2, z_2), \ldots, (x_n, y_n, z_n)\), and radii, \(\delta_1, \delta_2, \ldots, \delta_n\), shall be at the foot of one of these normals is \(\frac{n}{\pi m(\delta_1^2 + \delta_2^2 + \ldots + \delta_n^2)}\). And the probability that \(n\) points, taken at random on the surfaces of the spheres, shall be at the intersection of these normals with the line or surface is
\[\frac{n}{\pi m(\delta_1^2 + \delta_2^2 + \ldots + \delta_n^2)^n}\]
which probability is greatest when \(\delta_1^2 + \delta_2^2 + \ldots + \delta_n^2 = \delta_0^2\) is a minimum.

That is, from (1) and (2), the point, line or surface, which \(n\) points make the most probable, is the point line or surface which makes the sum of the squares of the normals upon it, or sum of the squares of the errors of situation a minimum.

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SOLUTIONS OF PROBLEMS IN NUMBER FIVE.

Solutions of problems in number five have been received as follows:

From Marcus Baker, 177, 178 and 179; Prof. W. P. Casey, 177 and 178; G. M. Day, 178; Prof. H. T. Eddy, 179; Edgar Frisby, 178; Newton Fitz, 175; Henry Gunder, 175, 177, 178, 179 and 180; Henry Heaton, 175, 177, 178, 179 and 180; Geo. Lillely, 179; Christine Ladd, 177; Prof. H. T. J. Ludwick, 178; Prof. D. J. Mc. Adam, 177, 178 and 179; Prof. Orson Pratt, 176; Werner Stille, 179; E. B. Seitz, 175, 177, 178, 179, 180, 181; Prof. J. Scheffer, 175, 177, 178, 179, 180; Prof. D. Trowbridge, 179.

175. "Find the roots of the equation \(x^4 + Ax^3 + Bx^2 + Cx + C^2 + A^2 = 0\)."

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

Multiplying the equation by \(4A^2\), then adding \((A^4 + 8AC - 4A^2B)x^3\) to both members, we get
\[4A^4x^4 + 4A^3x^3 + (A^4 + 8AC)x^2 + 4A^2Cx + 4C^2 = (A^4 + 8AC - 4A^2B)x^3.\]

Extracting the square root, we have
\[2Ax^2 + A^2x + 2C = \pm x\sqrt{(A^4 + 8AC - 4A^2B)}, \quad \text{whence}\]
\[x = \left\{ -A^2 \pm \sqrt{(A^4 + 8AC - 4A^2B)} \pm A \sqrt{2A^2 - 4B \pm 2\sqrt{(A^4 + 8AC - 4A^2B)}} \right\}/4A.\]
176. “Given \( u = F(y) \), and \( y = F'[z+xv f(y)] \);
also \( v = F_1(t) \), and \( t = F_2[z+xv f_1(t)] \),
to expand (by the differential calculus) \( u \) in a series of ascending, positive
and integral powers of \( x \). (\( x \) not being a function of \( z \)).”

**SOLUTION BY PROF. ORSON PRATT, SEN., SALT LAKE CITY, UTAH.**

We have \( u = F \{ F'[z+xv f(y)] \} = F''[z+xv f(y)] \);
also \( f(y) = f \{ F'[z+xv f(y)] \} = f' [z+xv f(y)] \).
We also have \( v = F_1 \{ F_2[z+xv f_1(t)] \} = F_3[z+xv f_1(t)] \);
and \( f_1(t) = f_1 \{ F_2[z+xv f_1(t)] \} = f_2[z+xv f_1(t)] \).

Let \( p = F''(z) ; q = f'(z) ; r = F_3(z) ; s = f_2(z) \). Then, by Laplace’s
theorem, we have

\[
\begin{align*}
u &= p + \frac{q dp}{dz} \frac{xv}{1} + d \frac{q^2 dp}{dz^2} \frac{x^2 v^2}{1.2} + d^2 \frac{q^3 dp}{dz^3} \frac{x^3 v^3}{1.2.3} + &c., \quad (1) \\
v &= r + \frac{sd r}{dz} \frac{xv}{1} + d \frac{s^2 dr}{dz^2} \frac{x^2 v^2}{1.2} + d^2 \frac{s^3 dr}{dz^3} \frac{x^3 v^3}{1.2.3} + &c. \quad (2)
\end{align*}
\]

By squaring (2), cubing, \&c., we have

\[
\begin{align*}
r^2 &= r^2 + \frac{sd(r^3)}{dz} \frac{xv}{1} + d \frac{s^2 d(r^3)}{dz^2} \frac{x^2 v^2}{1.2} + d^2 \frac{s^3 d(r^3)}{dz^3} \frac{x^3 v^3}{1.2.3} + &c., \quad (3) \\
v^3 &= r^3 + \frac{sd(r^3)}{dz} \frac{xv}{1} + d \frac{s^2 d(r^3)}{dz^2} \frac{x^2 v^2}{1.2} + d^2 \frac{s^3 d(r^3)}{dz^3} \frac{x^3 v^3}{1.2.3} + &c., \quad (4) \\
r^4 &= r^4 + \frac{sd(r^4)}{dz} \frac{xv}{1} + d \frac{s^2 d(r^4)}{dz^2} \frac{x^2 v^2}{1.2} + d^2 \frac{s^3 d(r^4)}{dz^3} \frac{x^3 v^3}{1.2.3} + &c., \quad (5)
\end{align*}
\]

Eliminate \( v, v^3, \&c. \), from (1), by substituting their values as determined
in (2), (3), \&c. In these substituted values again substitute for \( v, v^3, \&c. \).
Continue the process until \( v \) and its powers are excluded; after which add
the coefficients of the like powers of \( x \), and reduce their sums, and the re-
sult will be

\[
\frac{u = p}{1} + \frac{x}{1.2} \left( d \frac{q^2 dp}{dz^2} \cdot r^2 + \frac{q dp}{dz} \cdot s d(r^3) \right)
+ \frac{x^2}{1.2.3} \left( d^2 \frac{q^3 dp}{dz^3} \cdot r^3 + 2d \frac{q^2 dp}{dz} \cdot s d(r^3) + \frac{q dp}{dz} \cdot d^2 s d(r^3) \right)
\]
\[ x^4 + \frac{x^4}{1.2.3.4} \left( \frac{d^4 g^4 dp}{dx^4} \cdot r^4 + 3d^2 g^2 dp \cdot sd(r^4) \right) + 3d^2 g^2 dp \cdot d^2 s^2 d(r^4) + \frac{g dp}{dz} \cdot d^2 s^2 d(r^4) \right). \]

For the mth power of x, we have
\[
\frac{x^m}{1.2 \ldots m} \left( \frac{d^{m-1} g^{m-1} dp}{dz^{m-1}} \cdot r^m + (m-1) \frac{d^{m-2} g^{m-2} dp \cdot sd(r^m)}{dz^{m-1}} + \frac{(m-1)(m-2)}{1 \cdot 2} \right.
\]
\[
\times \frac{d^{m-3} g^{m-3} dp \cdot d^2 s^2 d(r^m)}{dz^{m-2}} + \ldots + \frac{g dp}{dz} \cdot d^{m-2} s^{m-1} d(r^m) \right). \]

By restoring the values of p, q, r and s, we have
\[ u = F''(z) \]
\[ + \frac{x}{1} \cdot f'(z) \cdot dF'(z) \cdot F_3(z) \]
\[ + \frac{x^2}{1.2} \cdot \left( \frac{d [f'(z)]^2 \cdot dF'(z) \cdot [F_3(z)]^a}{dz^2} + f'(z) \cdot dF'(z) \cdot f_2(z) \cdot d[F_3(z)]^a \right) \]
\[ + \frac{x^3}{1.2.3} \cdot (&c. \]  

This general theorem may be very much condensed, and clearly expressed in the following form:
\[ u = F''(z) + A^{(d^m)} x^m + A^{(d^1)} \frac{x^2}{1 \cdot 2} + A^{(d^3)} \frac{x^3}{1 \cdot 2 \cdot 3} + (&c. \]

(A)

In this, \( A^{(d^1)} \) is equal to the binomial differential coefficient of \( x^2 \); while
\( A^{(d^2)}, A^{(d^3)}, &c. \), represent the second, third, &c., differential expansions
of the binomial \( A^{(d^1)} \). By reference to the general term, it will be seen that
when \( m = 1 \), all the terms of the coefficient vanish, excepting the first; hence \( A^{(d^m)} \) reduces the binomial \( A^{(d^1)} \) to one term.

The celebrated differential theorem of Laplace is only a particular case of
the more general theorem (A). This will at once be seen, by making a
equal to unity in equation (1).

177. “If \( I_1, I_2, I_3 \) be the points of contact of the inscribed circle with
the sides of the triangle \( ABC, E_1, E_2, E_3 \) the centres of the
escribed circles, \( r \), the radius of the circle inscribed in \( I_1 I_2 I_3 \) and \( r \), the radius of that
inscribed in \( E_1 E_2 E_3 \), show that
\[ r = \frac{a + b + c}{a + b + r}, \quad r_2 = \frac{2R}{a + b + c}, \]
where \( R \) and \( r \) have their usual values and \( a, b, c \) are the distances between
the centres of the escribed circles.”
SOLUTION BY PROF. W. P. CASEY, SAN FRANCISCO, CAL.

Let \( ABC \) be the given triangle, \( E_1, E_2, E_3 \) the centres of the escribed circles, \( d \) the centre of the inscribed and \( O \) the centre of the circumscribed circle.

Now it is well known that the lines \( E_3 \), \( E_2 \), \( E_3E_1 \) and \( E_1E_2 \) are bisected by the circumference of the circle \( ABC \), or that the circle \( ABC \) is the nine point circle of \( E_1E_2E_3 \), and therefore that \( qE_1 \) or \( R_s = 2R \); but this can be shown without the nine point circle. Because \( E_1d = 2gm = 2E_1v \), and therefore \( gm \) is equal and parallel to \( E_1v \). Therefore \( E_1q = vm = 2R \). But \( E_1B, E_2A \) and \( E_3C \) are the perpendiculars of the triangle \( E_1E_2E_3 \), and by Ex. 226, page 320, Chauvenet's Geometry,

\[
R_s \times (a + b + c) = 2\Delta E_1E_2E_3, \quad 2R \times (a + b + c) = 2\Delta E_1E_2E_3; \quad \text{and} \quad r_s \times (a + \beta + \gamma) = 2\Delta E_1E_2E_3; \quad \text{and} \quad 2R(a + b + c) = r_s(a + \beta + \gamma), \quad \text{and hence}
\]

\[
r_s = 2R \cdot \frac{a + b + c}{a + \beta + \gamma}.
\]

Again, let \( r \) be the radius of the inscribed circle of the triangle \( I_1I_2I_3 \), and \( r \) that of the triangle \( ABC \) which is the circumscribed circle of the triangle \( I_1I_2I_3 \). It is plain that \( E_3E_2 \) is parallel to \( I_1I_2 \), and therefore \( I_1I_2I_3 \), \( E_1E_2E_3 \) are similar, whence we get the following equations:

\[
R_s \times (o + p + n) = r \times (a + \beta + \gamma) = 2R(o + p + n).
\]

Also, \( r_s(a + \beta + \gamma) = r_s(o + p + n) \). (2)

But from the preceding demonstration \( r_s(a + \beta + \gamma) = 2R(a + b + c) \), and

\[
2R(o + p + n) = r_s(a + \beta + \gamma), \quad \text{from (1).}
\]

Therefore by multiplying these two eq'ns we get

\[
r_s(o + p + n) = r_s(a + b + c) = r_s(a + \beta + \gamma), \quad \text{from (2). Hence}
\]

\[
r_s = r_s \cdot \frac{a + b + c}{a + \beta + \gamma}.
\]

Cor. \( 2R, r_s, r \) and \( r_s \) are in geometrical proportion; also \( \triangle E_1E_2E_3 \) and its circumscribed circle, the circle \( ABC \) and the nine point circle of the triangle \( ABC \) are in geometrical proportion.
The sides of the triangle \( I_1 I_2 I_3 \) are \( 2r \cos \frac{1}{2}A, \&c. \) Half the sum of the sides is then \( r (\cos \frac{1}{2}A + \cos \frac{1}{2}B + \cos \frac{1}{2}C) \).

The radius of the circle circumscribed about \( I_1 I_2 I_3 \) is \( r \). Substituting these values in the formula \( r = \frac{abc}{4Rs} \) we have

\[
r = \frac{8r^3 \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C}{4r^2 (\cos \frac{1}{2}A + \cos \frac{1}{2}B + \cos \frac{1}{2}C)}
\]

\[
r = \frac{2R (\cos \frac{1}{2}A + \cos \frac{1}{2}B + \cos \frac{1}{2}C)}{R}
\]

But it is known that \( a + b + c = 4R (\cos \frac{1}{2}A + \cos \frac{1}{2}B + \cos \frac{1}{2}C) \), hence

\[
r = \frac{a + b + c}{a + b + c}.
\]

It has been shown that the area of the triangle \( E_1 E_2 E_3 \) is \( 2Rs \); but the radius of the inscribed circle is equal to the area divided by half the perimeter, hence

\[
r = \frac{4Rs}{a + b + c} = 2R \frac{a + b + c}{a + b + c}.
\]

178. "Through any point \( O \) in a plane triangle \( ABC \) three lines \( a, b, \&c. \) are drawn parallel to the sides \( a, b \) and \( c \) respectively; prove that

\[
\frac{a}{a} + \frac{b}{b} + \frac{c}{c} = 2.
\]

SOLUTION BY EDGAR FRISBY, NAVAL OBS., WASH., D. C.

\[
a = \frac{AE}{c}, \text{ by similar triangles; also } b = \frac{BD}{c}, \ " \ " \ " \ , \text{ and } c = \frac{BE + AD}{c}, \text{ from opposite sides of parallelograms } BO \text{ and } AO.
\]

By addition we have

\[
\frac{a + b + c}{a + b + c} = \frac{(AE + BE) + (BD + AD)}{c} = 2c + c = 2.
\]

179. "Integrate \( \frac{\sin \theta d\theta}{(\sin \theta + \cos \theta)^8} \)"
SOLUTION BY MARCUS BAKER, U. S. COAST SURVEY, WASHINGTON, D. C.

\[
\int \frac{\sin \theta \, d\theta}{(\sin \theta + \cos \theta)^3} = \int \frac{d\theta}{(\sin \theta + \cos \theta)^3} - \int \frac{\cos \theta \, d\theta}{(\sin \theta + \cos \theta)^2} \\
= \int \frac{\sec \theta \, d\theta}{(1 + \tan \theta)^2} - \int \frac{\sec^3 \theta \, d\theta}{(1 + \tan \theta)^3} \\
= \frac{1}{2(1 + \tan \theta)^2} - \frac{1}{1 + \tan \theta} + C.
\]

SOLUTION BY GEO. LILLEY, KEWANEE, ILL.

Let \(u = \int \frac{\sin \theta \, d\theta}{(\sin \theta + \cos \theta)^3} = \int \frac{\tan \theta \sec \theta \, d\theta}{(\tan \theta + 1)^3} \), and put \(x = \tan \theta\); then

\[u = \int \frac{xdx}{(x + 1)^3}.
\]

Put \((x+1)^3 = v\), then \(u = \frac{1}{3} \int \frac{e^v - 1}{\sqrt{v}} dv = \frac{1 - 2e^v}{2\sqrt{v}} = \frac{1 + 2\tan \theta}{2(1 + \tan \theta)^3} + C.
\]

180. "Two equal spheres placed in a paraboloid with its axis vertical touch one another at the focus. If \(W\) be the weight of a sphere, \(R, R'\) the pressures upon it, prove that \(W^2 : R : R' :: 3 : 2\)."

SOLUTION BY HENRY HEATON, SABULA, IOWA.

Let \(O\) and \(O'\) be the centers of the two spheres, \(F\) the focus, \(CD\) the axis, and \(ACB\) a section of the surface of the paraboloid, the spheres being tangent to the surface at \(A\) and \(B\).

\(AD\) and \(BD\) drawn through \(O\) and \(O'\) are normals to \(ACB\). Put \(AE = y, CE = x\) and \(ED\), the subnormal, = \(p\).

Then \(y^2 = 2px\), and \(FE = \frac{1}{2}y - x = \frac{p^2 - y^2}{2p}. \quad \angle DAE = 2 \angle FAE. \quad 
\]

\[\tan^{-1} \left( \frac{y}{x} \right) = 2 \tan^{-1} \left( \frac{p^2 - y^2}{2py} \right) = \tan^{-1} \left( \frac{4py(p^2 - y^2)}{-y^2 + 6p^2y^2 - p^4} \right). \]

Therefore \(p = \frac{4py(p^2 - y^2)}{-y^2 + 6p^2y^2 - p^4}\), and \(p^3 = 3y^2. \quad \therefore AD = 2y. \]

Since the pressures \(R\) and \(R'\) are in the directions of \(AD\) and \(AE\), and the weight acts parallel to \(DE\), \(R : R' : W :: 2y : y : y\sqrt{3} :: 2 : 1 : \sqrt{3}. \]

Therefore \(\frac{1}{2} R = \frac{1}{2} R' = \frac{W}{\sqrt{3}}; \quad \therefore \frac{1}{2} RR' = \frac{1}{2} W^2\); or \(W^2 : RR' :: 3 : 2.\)
181 “Find the volume between \( x = 0 \) and \( x = 2l \) of the solid bounded by the surface whose equation is
\[
a(y^2+z^2) - x(y^2-2xy^2+2x^2) - y^2(bx^3+c^2x+e^2) = 0.
\]

SOLUTION BY E. R. SEITZ.

Let \( y = r \sin \theta \) and \( z = r \cos \theta \). Then the equation becomes
\[
ar^2 = (x^2+2e^2)\cos^2\theta + (bx^3+c^2x+e^3)\sin^2\theta,
\]
and
\[
V = \int \int \int r dr d\theta dx.
\]
The limits of \( r \) are 0 and \( \left( (x^3+2e^3)\cos^2\theta + (bx^3+c^2x+e^3)\sin^2\theta \right) \alpha \), of \( \theta \), 0 and \( 2\pi \); and of \( x, 0 \) and \( 2l \).

\[
V = \int_0^2 \int_0^{2\pi} \int_0^r r dr d\theta dx = \frac{1}{4a} \int_0^{2\pi} \int_0^{2\pi} \left[ x^3 + bx^3 + c^2x + 3e^3 + (x^3-bx^3-c^2x+e^3)\cos 2\theta \right] d\theta dx
\]
\[
= \frac{\pi}{2a} \int_0^{2\pi} (x^3 + bx^3 + c^2x + 3e^3) dx = \frac{\pi l}{3a} \left( 6l^3 + 4bl^3 + 3c^2l + 9e^3 \right).
\]

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Note by the Editor. — In the Note on Attraction, at page 181, we modified Mr. Adcock's language, supposing that we had retained his ideas (except in one particular, which was thought to be a mistake), and sent him a copy of the "proof" for his approval or rejection. But, unfortunately, the proof was not returned until after the sheet was printed. Therefore, as Mr. Adcock does not accept the modification, we subjoin his note, verbatim.

"Ultimate Proposition in Attraction. — If the points of every particle of matter attract the points of every other particle of matter, then the resultant attraction between any two particles, whose dimensions are infinitely small in comparison with the distance between them, will be directly as the product of their masses and inversely as the square of the distance between them.

"That it is as the product of the masses results from the consideration that if the densities be increased or diminished in any ratio the number of attracting points or forces is increased or diminished in the same ratio without affecting the directions.

"That it is in the inverse ratio of the square of the distance follows, first, from the fact that any two points, of matter of the same density, attract each other with a constant force for all distances because the point of application of a force may be anywhere on its line of direction, distance being a quantity of a different kind has no effect on force; second, each point of
each particle being a centre of attraction for all other points of all particles, the number of these equal forces applied between two particles, is directly as the product of their masses and inversely as the square of the distance between them, as a necessary consequence of their situation.

R. J. Adcock."

We embrace this opportunity to say, that we do not reject papers submitted for publication by mathematicians of acknowledged ability simply because we dissent from the conclusions arrived at, or the logic employed, nor do we always signify our dissent, as it is assumed that a majority of our readers are quite as competent to decide the matter as we are.

In relation to the above note by Mr. Adcock, we have to say, that we do not understand what is meant by the points of a particle infinitely small, nor do we understand how the density of a particle infinitely small can be increased or diminished. And that the attraction between masses is as the \textit{sum} of the masses and not as their \textit{product} we supposed to be a universally admitted fact.

\textbf{PROBLEMS.}

182. \textbf{(Selected)} By R. M. De France, Esq.—If \( \tan \theta + \sin \theta = m \) and \( \tan \theta - \sin \theta = n \), show that \( m^2 - n^2 = 4 \sqrt{(mn)} \).

183. \textbf{(Selected)} By Prof. Eddy.—Show that the altitude of the greatest equilateral triangle that can be circumscribed about a given triangle, is \( \sqrt{[a^2 + b^2 - 2ab \cos (\frac{\pi}{3} + C)]} \).

184. By O. H. Merrill.—In cutting the maximum rectangular parallelopipedon from a frustrum of a cone, five pieces are cut off. Find the volume of each of these pieces.

185. By Prof. J. J. Skinner, New Haven, Conn.—Let the equation \( ay^2 + bxy + cx^2 + dy + ex + f = 0 \) represent a parabola referred to rectangular axes. Prove that the latus rectum \( = (bd - 2ac) \div [(a + c) \sqrt{(4a^2 + b^3)}] \).

186. By Prof. Johnson.—\( A_1, A_2, B_1, B_2, \) and \( C_1, C_2 \) are three pairs of points in a plane such that the three lines \( A_1 A_2, B_1 B_2, C_1 C_2 \) meet in a common point \( O \). Let \( B_1 C_1 \) and \( B_2 C_2 \) meet in \( a \), and determine in like manner the points \( b \) and \( c \). Prove that \( a, b \) and \( c \) are in a straight line.

\textbf{Note.} We have thus 10 points situated 3 by 3 on 10 lines, each point being the intersection of 3 lines, so that for each point there are three pairs of points colinear with it, and three points not joined to it in the figure but colinear with each other. Devise a notation which shall express the mutual relation of these points and lines.
ANNOUNCEMENT. — From the many kind and encouraging letters rec'd in response to the Special Notice, published in No. 5, we do not hesitate to announce the continuance of the Analyst.

We have no additional inducements to offer for the purpose of obtaining new subscribers, but, bearing in mind that each additional subscriber will enhance the value of the Journal, by increasing the number of its contributors and the variety of its contents, as well as contribute to its life and support, it is hoped that its friends will, each, make such reasonable effort to extend its circulation as circumstances may justify.

We shall endeavor, in the future as in the past, to make the Analyst as acceptable to its readers and contributors as our ability and circumstances will permit; having made definite arrangements for its continuance at least two years longer.

No. 1, Vol. V, will be issued about the 25th of Dec., and will contain a paper on the Grouping of Signs of Residuals, by De Forest, the conclusion of Mr. Meech's paper on Elliptic Functions, and other articles of interest to mathematicians, besides the usual number of problems and solutions.

Editor.

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PUBLICATIONS RECEIVED.


*Transactions of the Wisconsin Academy of Science, Arts and Letters.* 8vo. 269 pages. Madison, Wisconsin. 1876. — This publication contains many valuable papers, the most interesting of which, perhaps, to mathematicians, is a paper on Recent Progress in Theoretical Physics, by John E. Davies, A. M., M. D., Professor of Physics in the Univ. of Wisconsin.

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ERRATA.

On page 153, line 3, for “and the edges by (01)” &c., read, and the squares of the edges &c.

" 156, " 13, for $\frac{1}{2}$, read $\frac{1}{2}$. And on page 162, line 2, for $\theta$, read $\phi$.

" 162, " 5, for $\sqrt{\left(\cos \phi +1\right) \cdot \sqrt{\cos \frac{\phi}{2}}} \cdot \sqrt{\cos \frac{\phi}{2}}$, read $(1+\sqrt{\cos \phi})\sqrt{\cos \frac{\phi}{2}}$.

" 22, " 22, insert a bracket immediately after the radical sign in the denominator of the last member of the equation, and a corresponding bracket at the end of the line.

On page 166, line 8, change the three $+$ signs to $-$, and for $\frac{1}{2}e^4$, read $\frac{1}{2}e^8$.

" 181, " 3, for additions, read additional.

" 282, " 11, and line 16, for (17) and (16), read respectively (14) and (13).

" 8 from bottom, $d e$ — .
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| Garnets,                | Sardonyx           | Bismuth Silver,|
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| Alston Moor Spar,       | Tourmaline         | Antimonial Silver, |
| Dog Tooth Spar,         | Epidote            | Galena,        |
| Calc Spar,              | Lapis Lazuli       | Zinc Blende, |
| Iceland Spar,           | Iron Pyrites       | Geysersite,    |
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JESSE S. RANDALL,
Georgetown, Colorado