and therefore 
\[ \tan \alpha \int_{v_{1}}^{v_{2}} \frac{\Delta(v) \cot v}{1 + \cot^2 \alpha \sin^2 v} \, dv = - \int_{v_{1}}^{v_{2}} \frac{\Delta(v) \tan \alpha}{1 + \cot^2 \alpha \sin^2 v} \, dv - K' - K_s \]
\[ = - K' - K_s - K'_s \]
say. The equation therefore becomes
\[ K' - K_s - K_s = - K' - K_s - K'_s \]
that is 
\[ K' + K'_s = 0. \]
This is
\[ \int_{v_{1}}^{v_{2}} \frac{\Delta(v)}{1 + \cot^2 \alpha \sin^2 v} \left\{ \cot \alpha + \frac{\tan \alpha}{\sin^2 v} \right\} \, dv = 0; \]
or, neglecting the factor \( \tan \alpha \), we have the equation in the form
\[ \int_{v_{1}}^{v_{2}} \frac{\Delta(v)}{\sin^2 v} \, dv = 0. \]

5. To evaluate this integral, we now introduce elliptic functions as in the earlier paper. Writing
\[ v = \alpha m \, u, \]
and denoting by \( u_s \) and \( u_i \) the limits corresponding to \( v_s \) and \( v_i \), we find
\[ \int_{u_s}^{u_i} \frac{dn'du}{nu} = 0. \]
But
\[ \int_{nu_s}^{nu_i} \frac{dn'}{nu} = k^2 n - E(u) - \frac{\text{cn} u \, dn u}{\text{sn} u}, \]
as an indefinite integral; and therefore we have
\[ k^2 u_i - E(u_i) - \frac{\text{cn} u_i \, dn u_i}{\text{sn} u_i} = k^2 u_s - E(u_s) - \frac{\text{cn} u_s \, dn u_s}{\text{sn} u_s}, \]
as the equation determining \( u_i \) when \( u_s \) is given.

One remark may be made about this result. Taking
\[ \int_{nu_s}^{nu_i} \frac{dn'}{nu} = f(u), \]
we have \( df(u)/du \) a positive quantity, so that \( f(u) \) is continually increasing as \( u \) increases; yet the equation gives \( f(u) = f(u_s) \). The explanation is that \( f(u) \) passes through an infinite value—at \( u = 2K \)—and changes its sign from positive to negative; it then begins to increase again from \(-\infty\).

The equation can be taken in the form
\[ k^2 (u_i - u_s) = E(u_i) - E(u_s) - \frac{\text{cn} u_i \, dn u_i}{\text{sn} u_i} - \frac{\text{cn} u_s \, dn u_s}{\text{sn} u_s} \]
\[ = E(u_i - u_s) - k^2 \, \text{sn} u_s \, \text{sn} u_i \, \text{sn} (u_i - u_s) \]
\[ - \frac{\text{sn} u_i \, \text{sn} u_s \, (1 - k^2 \, \text{sn}^2 u_i \, \text{sn}^2 u_s)}{\text{sn} u_i \, \text{sn} u_s}, \]
and therefore finally
\[ \text{sn} (u_i - u_s) = \{E(u_i - u_s) - k^2 (u_i - u_s) \} \, \text{sn} u_i \, \text{sn} u_s, \]
is the equation to determine \( u_i \). It is evidently satisfied by \( u_i = u_s \); the required value of \( u_i \) is the root of the equation next greater than \( u_s \).
If preferred, the Zeta-function can be used instead of the second elliptic integral. For any argument, we have
\[ Z(u) = E(u) - \frac{E}{K} \, u, \]
and, therefore,
\[ E(u) - k^2 \, u = Z(u) + \frac{E - k^2 \, K}{K} \, u \]
\[ = Z(u) + \frac{G}{K} \, u, \]
where \( G \) is the quantity introduced by Glaisher. The equation thus becomes
\[ \text{sn} (u_i - u_s) = E(u_i - u_s) - k^2 (u_i - u_s) \]
\[ = Z(u_i - u_s) + \frac{G}{K} (u_i - u_s) \]
\[ = Z(u_i - u_s) + \frac{2\zeta' \, dK}{d\zeta} (u_i - u_s). \]

6. Taking the equation in the form
\[ \text{sn}(u_1 - u_o) = \text{sn} u \text{ sn} u_o \left[ E(u_1 - u_o) - c' (u_1 - u_o) \right], \]
we at once see that

- if \( u_o = 0 \), then \( u_1 = 2K \),
- and if \( u_o = 2K \), then \( u_1 = 4K \).

On account of the symmetrical undulation of the geodesic, it is sufficient to take into account values of \( u_o \), such that

\[ 0 < u_o < 2K, \]
in addition to the two limiting values already mentioned.

If we consider the curves,

\[ y_1 = \frac{\text{sn}(x - u_o)}{\text{sn} x \text{ sn} u_o}, \quad y_1 = E(x - u_o) - c' (x - u_o), \]
the first value of \( x \) increasing from \( u_o \), for which \( y_1 = y_2 \) is \( u_1 \)

![Fig 1](image1)

![Fig 2](image2)

Now these curves, when traced, are as in figures 1 and 2 respectively; hence

\[ u_o + 2K < u_1 < 4K. \]

If we take \( u_1 - u_o = 2K + \theta' \),
so that \( 0 < \theta' < 2K - u_o \),
we have

\[ y_1 = \frac{\text{sn}(2K + \theta') \text{ sn} u_o}{\text{sn} \theta' \text{ sn} u_o}, \]

\[ y_1 = 2G + E(\theta') - c' \theta', \]
and \( \theta' \) is determined, between the above limits by the equation

\[ \frac{\text{sn} \theta'}{\text{sn} (u_0 + \theta')} \text{ sn} u_o = 2G + E(\theta') - c' \theta'. \]

7. When the ellipticity of the oblate spheroid is small, \( c \) is small. Then \( G \) is small; and as

\[ E(\theta') - c' \theta' = c \int_0^{\theta'} \text{sn} u du, \]
it follows that \( \text{sn} \theta' \) (and therefore \( \theta' \)) is small of the same order as \( c \). Consequently \( E(\theta') - c' \theta' \) is of the second order in \( c \); also

\[ G = 2c \frac{dK}{dc} = \frac{1}{2} \pi c, \]
so far as regards the most important term. Hence, when second powers of the ellipticity (and therefore second powers of \( c \)) are negligible, we have

\[ \theta' = \frac{1}{2} \pi c \text{ sn} u_o, \]
and consequently the excess of \( u_1 - u_o \) above \( \pi \) is

\[ \frac{1}{2} \pi c \left(1 + 2 \text{ sn}^2 u_o\right) \]
to the first order of small quantities.

8. In what precedes, it has been assumed that the geodesic is not a meridian. The special case in which it is a meridian is given by \( \alpha = 0 \) and then \( k^2 = c' \); so that the preceding formulae apply on the understanding that the modulus of the elliptic functions is \( c' \) instead of

\[ \frac{c' \cos^2 \alpha}{1 - c' \sin^2 \alpha}. \]

In particular, when \( u_o = 0 \) the conjugate is given by

\[ u_1 = 2\pi \text{ or } \theta = \pi: \text{ that is, the conjugate of either pole is the other pole—a result to be expected.} \]