and therefore

$$\tan\alpha\int_{v_0}^{v_1}\frac{d}{dv}\left\{\frac{\Delta(v)\cot v}{1+\cot^2\alpha\sin^2 v}\right\}dv$$

$$=-\int_{v_0}^{v_1}\frac{\Delta(v)\tan\alpha}{(1+\cot^2\alpha\sin^2 v)\sin^2 v}\,dv-K_2-K_3$$

$$=-K_1'-K_2-K_3,$$

say. The equation therefore becomes

$$K_1-K_2-K_3=-K_1'-K_2-K_3,$$

that is

$$K_1+K_1'=0.$$

This is

$$\int_{v_0}^{v_1}\frac{\Delta(v)}{1+\cot^2\alpha\sin^2 v}\left\{\cot\alpha+\frac{\tan\alpha}{\sin^2 v}\right\}dv=0;$$

or, neglecting the factor $\tan\alpha$, we have the equation in the form

$$\int_{v_0}^{v_1}\frac{\Delta(v)}{\sin^2 v}\,dv=0.$$

5. To evaluate this integral, we now introduce elliptic functions as in the earlier paper. Writing

$$v=\operatorname{am}u,$$

and denoting by u_0 and u_1 the limits corresponding to v_0 and v_1, we find

$$\int_{u_0}^{u_1}\frac{\operatorname{dn}^2 u}{\operatorname{sn}^2 u}\,du=0.$$

But

$$\int\frac{\operatorname{dn}^2 u}{\operatorname{sn}^2 u}\,du=k'^2u-E(u)-\frac{\operatorname{cn}u\operatorname{dn}u}{\operatorname{sn}u},$$

as an indefinite integral; and therefore we have

$$k'^2u_1-E(u_1)-\frac{\operatorname{cn}u_1\operatorname{dn}u_1}{\operatorname{sn}u_1}=k'^2u_0-E(u_0)-\frac{\operatorname{cn}u_0\operatorname{dn}u_0}{\operatorname{sn}u_0},$$

as the equation determining u_1 when u_0 is given.

One remark may be made about this result. Taking

$$\int\frac{\operatorname{dn}^2 u}{\operatorname{sn}^2 u}\,du=f(u),$$

we have $df(u)/du$ a positive quantity, so that $f(u)$ is continually increasing as u increases; yet the equation gives $f(u_1)=f(u_0)$. The explanation is that $f(u)$ passes through an infinite value—at $u=2K$—and changes its sign from positive to negative; it then begins to increase again from $-\infty$.

The equation can be taken in the form

$$k'^2(u_1-u_0)=E(u_1)-E(u_0)-\left(\frac{\operatorname{cn}u_0\operatorname{dn}u_0}{\operatorname{sn}u_0}-\frac{\operatorname{cn}u_1\operatorname{dn}u_1}{\operatorname{sn}u_1}\right)$$

$$=E(u_1-u_0)-k^2\operatorname{sn}u_1\operatorname{sn}u_0\operatorname{sn}(u_1-u_0)$$

$$-\frac{\operatorname{sn}(u_1-u_0)}{\operatorname{sn}u_1\operatorname{sn}u_0}(1-k^2\operatorname{sn}^2u_1\operatorname{sn}^2u_0)$$

$$=E(u_1-u_0)-\frac{\operatorname{sn}(u_1-u_0)}{\operatorname{sn}u_1\operatorname{sn}u_0};$$

and therefore finally

$$\operatorname{sn}(u_1-u_0)=\{E(u_1-u_0)-k'^2(u_1-u_0)\}\operatorname{sn}u_1\operatorname{sn}u_0$$

is the equation to determine u_1. It is evidently satisfied by $u_1=u_0$; the required value of u_1 is the root of the equation next greater than u_0.

If preferred, the Zeta-function can be used instead of the second elliptic integral. For any argument, we have

$$Z(u)=E(u)-\frac{E}{K}u,$$

and, therefore,

$$E(u)-k'^2u=Z(u)+\frac{E-k'^2K}{K}u$$

$$=Z(u)+\frac{G}{K}u,$$

where G is the quantity introduced by Glaisher.* The equation thus becomes

$$\frac{\operatorname{sn}(u_1-u_0)}{\operatorname{sn}u_1\operatorname{sn}u_0}=E(u_1-u_0)-k'^2(u_1-u_0)$$

$$=Z(u_1-u_0)+\frac{G}{K}(u_1-u_0)$$

$$=Z(u_1-u_0)+\frac{2cc'}{K}\frac{dK}{dc}(u_1-u_0).$$

* *Quart. Journal*, Vol. XIX. (1883), pp. 145—157; *ib.*, Vol. XX. (1886), pp. 313—361.

6. Taking the equation in the form

$$\operatorname{sn}(u_1 - u_0) = \operatorname{sn} u_1 \operatorname{sn} u_0 \{E(u_1 - u_0) - c'(u_1 - u_0)\},$$

we at once see that

if $\quad u_0 = 0, \quad$ then $u_1 = 2K,$

and if $\quad u_0 = 2K, \quad$ then $u_1 = 4K.$

On account of the symmetrical undulation of the geodesic, it is sufficient to take into account values of u_0, such that

$$0 < u_0 < 2K,$$

in addition to the two limiting values already mentioned.

If we consider the curves,

$$y_1 = \frac{\operatorname{sn}(x - u_0)}{\operatorname{sn} x \operatorname{sn} u_0}, \quad y_2 = E(x - u_0) - c'(x - u_0),$$

the first value of x increasing from u_0, for which $y_1 = y_2$, is u_1

Fig. 1

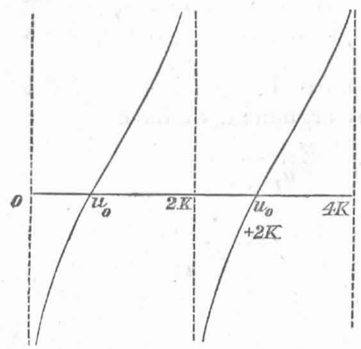
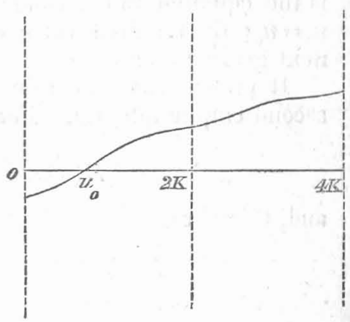


Fig. 2



Now these curves, when traced, are as in figures 1 and 2 respectively; hence

$$u_0 + 2K < u_1 < 4K.$$

If we take $\quad u_1 - u_0 = 2K + \theta_1,$

so that $\quad 0 < \theta_1 < 2K - u_0,$

we have
$$y_1 = \frac{\operatorname{sn}(2K + \theta_1)}{\operatorname{sn}(2K + u_0 + \theta_1) \operatorname{sn} u_0}$$
$$= \frac{\operatorname{sn} \theta_1}{\operatorname{sn}(u_0 + \theta_1) \operatorname{sn} u_0},$$
$$y_2 = 2G + E(\theta_1) - c'\theta_1;$$

and θ_1 is determined, between the above limits by the equation

$$\frac{\operatorname{sn} \theta_1}{\operatorname{sn}(u_0 + \theta_1) \operatorname{sn} u_0} = 2G + E(\theta_1) - c'\theta_1.$$

7. When the ellipticity of the oblate spheroid is small, c is small. Then G is small; and as

$$E(\theta_1) - c'\theta_1 = c \int_0^{\theta_1} \operatorname{cn}^2 u \, du,$$

it follows that $\operatorname{sn} \theta_1$ (and therefore θ_1) is small of the same order as c. Consequently $E(\theta_1) - c'\theta_1$ is of the second order in c; also

$$G = 2cc' \frac{dK}{dc}$$
$$= \tfrac{1}{4}\pi c,$$

so far as regards the most important term. Hence, when second powers of the ellipticity (and therefore second powers of c) are negligible, we have

$$\theta_1 = \tfrac{1}{2}\pi c \operatorname{sn}^2 u_0,$$

and consequently the excess of $u_1 - u_0$ above π is

$$\tfrac{1}{4}\pi c\,(1 + 2 \operatorname{sn}^2 u_0)$$

to the first order of small quantities.

8. In what precedes, it has been assumed that the geodesic is not a meridian. The special case in which it is a meridian is given by $\alpha = 0$ and then $k^2 = e^2$; so that the preceding formulæ apply on the understanding that the modulus of the elliptic functions is e^2 instead of

$$\frac{e^2 \cos^2\alpha}{1 - e^2 \sin^2\alpha}.$$

In particular, when $u_0 = 0$ the conjugate is given by $u_1 = 2\pi$ or $\theta = \pi$: that is, the conjugate of either pole is the other pole—a result to be expected.

25 *January*, 1896.